NONADDITIVE ENTROPIES – FOUNDATIONS AND APPLICATIONS IN PHYSICS AND ELSEWHERE

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Foundations of Statistical Mechanics

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The first man to use a truly statistical approach was Boltzmann [1877] and at that point kinetic theory changed into statistical mechanics even though it was another twenty odd years before Gibbs coined the expression.

One might argue that the proof of the pudding is in the eating, and that the fact that statistical mechanics has been able to predict accurately and successfully the behavior of physical systems under equilibrium conditions—and even under certain circumstances in non-equilibrium conditions⁴—should be a sufficient justification for the methods used.
The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts. This is true if the energy of the system is the sum of the energies of all the parts and if the work performed by the system during a transformation is equal to the sum of the amounts of work performed by all the parts. Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled. Thus, for example, in the case of a system composed of two homogeneous substances, it will be possible to express the energy as the sum of the energies of the two substances only if we can neglect the surface energy of the two substances where they are in contact. The surface energy can generally be neglected only if the two substances are not very finely subdivided; otherwise, it can play a considerable role.
**ENTROPIC FUNCTIONALS**

<table>
<thead>
<tr>
<th>BG entropy (q = 1)</th>
<th>Entropy Sq (q real)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_i = \frac{1}{W}$ (all $i$)</td>
<td>Equiprobability</td>
</tr>
<tr>
<td>$\forall p_i (0 \leq p_i \leq 1)$</td>
<td>Extensive</td>
</tr>
<tr>
<td>$\left( \sum_{i=1}^{W} p_i = 1 \right)$</td>
<td>Concave</td>
</tr>
<tr>
<td>$k \ln W$</td>
<td>Lesche-stable</td>
</tr>
<tr>
<td>$-k \sum_{i=1}^{W} p_i \ln p_i$</td>
<td>Finite entropy production per unit time</td>
</tr>
</tbody>
</table>

- **additive**
- **nonadditive (if $q \neq 1$)**
- **Composable**
- **Topsoe-factorizable (unique)**
- **Amari-Ohara-Matsuzoe conformally invariant geometry (unique)**
- **Biro-Barnafoldi-Van thermostat universal independence (unique)**
**DEFINITIONS**: $q$-logarithm: \[ \ln_q x \equiv \frac{x^{1-q} - 1}{1-q} \quad (x > 0; \quad \ln_1 x = \ln x) \]

$q$-exponential: \[ e_q^x \equiv [1 + (1 - q) x]^{\frac{1}{1-q}} \quad (e_1^x = e^x) \]

Hence, the entropies can be rewritten:

<table>
<thead>
<tr>
<th></th>
<th>equal probabilities</th>
<th>generic probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BG entropy</strong></td>
<td>$k \ln W$</td>
<td>$k \sum_{i=1}^{W} p_i \ln \frac{1}{p_i}$</td>
</tr>
<tr>
<td>$(q = 1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>entropy $S_q$</strong></td>
<td>$k \ln_q W$</td>
<td>$k \sum_{i=1}^{W} p_i \ln_q \frac{1}{p_i}$</td>
</tr>
<tr>
<td>$(q \in \mathbb{R})$</td>
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</tbody>
</table>
TYPICAL SIMPLE SYSTEMS:

- Short-range space-time correlations
- Markovian processes (short memory), Additive noise
- Strong chaos (positive maximal Lyapunov exponent), Ergodic, Riemannian geometry
- Short-range many-body interactions, weakly quantum-entangled subsystems
- Linear and homogeneous Fokker-Planck equations, Gaussians
  - → Boltzmann-Gibbs entropy (additive)
  - → Exponential dependences (Boltzmann-Gibbs weight, ...)

TYPICAL COMPLEX SYSTEMS:

- Long-range space-time correlations
- Non-Markovian processes (long memory), Additive and multiplicative noises
- Weak chaos (zero maximal Lyapunov exponent), Nonergodic, Multifractal geometry
- Long-range many-body interactions, strongly quantum-entangled subsystems
- Nonlinear and/or inhomogeneous Fokker-Planck equations, $q$-Gaussian
  - → Entropy $S_q$ (nonadditive)
  - → $q$-exponential dependences (asymptotic power-laws)

An entropy is **additive** if, for any two probabilistically independent systems $A$ and $B$,

$$S(A + B) = S(A) + S(B)$$

Therefore, since

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q) S_q(A) S_q(B),$$

$S_{BG}$ and $S_q^{Renyi}$ ($\forall q$) are additive, and $S_q$ ($\forall q \neq 1$) is nonadditive.

**EXTENSIVITY:**

Consider a system $\Sigma \equiv A_1 + A_2 + ... + A_N$ made of $N$ (not necessarily independent) identical elements or subsystems $A_1$ and $A_2$, ..., $A_N$.

An entropy is **extensive** if

$$0 < \lim_{N \to \infty} \frac{S(N)}{N} < \infty, \ i.e., \ S(N) \propto N \ (N \to \infty)$$
EXTENSIVITY OF THE ENTROPY \((N \rightarrow \infty)\)

If \(W(N) \sim \mu^N\) \((\mu > 1)\)

\[\Rightarrow S_{BG}(N) = k_B \ln W(N) \propto N \quad \text{OK!}\]

If \(W(N) \sim N^\rho\) \((\rho > 0)\)

\[\Rightarrow S_q(N) = k_B \ln_q W(N) \propto [W(N)]^{1-q} \propto N^{\rho(1-q)}\]

\[\Rightarrow S_{q=1-1/\rho}(N) \propto N \quad \text{OK!}\]

If \(W(N) \sim \nu^{N^\gamma}\) \((\nu > 1; \ 0 < \gamma < 1)\)

\[\Rightarrow S_\delta(N) = k_B \left[ \ln W(N) \right]^\delta \propto N^{\gamma \delta}\]

\[\Rightarrow S_{\delta=1/\gamma}(N) \propto N \quad \text{OK!}\]

IMPORTANT: \(\mu^N \gg \nu^{N^\gamma} \gg N^\rho \quad \text{if} \quad N \gg 1\)
Nonadditive entropy reconciles the area law in quantum systems with classical thermodynamics

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The Boltzmann–Gibbs–von Neumann entropy of a large part (of linear size $L$) of some (much larger) $d$-dimensional quantum systems follows the so-called area law (as for black holes), i.e., it is proportional to $L^{d-1}$. Here we show, for $d=1,2$, that the (nonadditive) entropy $S_q$ satisfies, for a special value of $q \neq 1$, the classical thermodynamical prescription for the entropy to be extensive, i.e., $S_q \propto L^d$. Therefore, we reconcile with classical thermodynamics the area law widespread in quantum systems. Recently, a similar behavior was exhibited in mathematical models with scale-invariant correlations [C. Tsallis, M. Gell-Mann, and Y. Sato, Proc. Natl. Acad. Sci. U.S.A. 102 15377 (2005)]. Finally, we find that the system critical features are marked by a maximum of the special entropic index $q$. 
Block entropy for the $d=1+1$ model, with central charge $c$, at its quantum phase transition at $T=0$ and critical transverse “magnetic” field analytically obtained from first principles.

Self-dual $Z(n)$ magnet ($n=1,2,...$) [FC Alcaraz, JPA 20 (1987) 2511]

$$c = \frac{2(n-1)}{n+2} \in [0,2]$$

$SU(n)$ magnets ($n=1,2,...; m=2,3,...$) [FC Alcaraz and MJ Martins, JPA 23 (1990) L1079]

$$c = (n-1) \left[ 1 - \frac{n(n+1)}{(m+n-2)(m+n-1)} \right] \in [0,n-1]$$
Thermostatistics of Overdamped Motion of Interacting Particles

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We show through a nonlinear Fokker-Planck formalism, and confirm by molecular dynamics simulations, that the overdamped motion of interacting particles at $T = 0$, where $T$ is the temperature of a thermal bath connected to the system, can be directly associated with Tsallis thermostatistics. For sufficiently high values of $T$, the distribution of particles becomes Gaussian, so that the classical Boltzmann-Gibbs behavior is recovered. For intermediate temperatures of the thermal bath, the system displays a mixed behavior that follows a novel type of thermostatistics, where the entropy is given by a linear combination of Tsallis and Boltzmann-Gibbs entropies.
See also:
Levin and Pakter, PRL 107, 088901 (2011)
Andrade, Silva, Moreira, Nobre and Curado, PRL 107, 088902 (2011)
Ribeiro, Nobre and Curado, PRE 85, 121046 (2012)
Nobre, Souza and Curado, PRE 86, 061113 (2012)
Curado, Souza, Nobre and Andrade, PRE 89, 022117 (2014)
Andrade, Souza, Curado and Nobre, EPL 108, 20001 (2014)

Group entropies, correlation laws, and zeta functions

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(Received 15 February 2011; revised manuscript received 3 May 2011; published 10 August 2011)

The notion of group entropy is proposed. It enables the unification and generalization of many different definitions of entropy known in the literature, such as those of Boltzmann-Gibbs, Tsallis, Abe, and Kaniadakis. Other entropic functionals are introduced, related to nontrivial correlation laws characterizing universality classes of systems out of equilibrium when the dynamics is weakly chaotic. The associated thermostatistics are discussed. The mathematical structure underlying our construction is that of formal group theory, which provides the general structure of the correlations among particles and dictates the associated entropic functionals. As an example of application, the role of group entropies in information theory is illustrated and generalizations of the Kullback-Leibler divergence are proposed. A new connection between statistical mechanics and zeta functions is established. In particular, Tsallis entropy is related to the classical Riemann zeta function.

\[ S_q \leftrightarrow \frac{1}{(1-q)^{s-1}} \zeta(s) \quad (q < 1) \]

with \( \zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \)

\[ = \frac{1}{1-2^{-s}} \frac{1}{1-3^{-s}} \frac{1}{1-5^{-s}} \frac{1}{1-7^{-s}} \frac{1}{1-11^{-s}} \ldots \]
The relation between $S$ and $W$ given in Eq. (1) is the only reasonable given the proposition that the entropy of a system consisting of subsystems is equal to the sum of entropies of the subsystems.

(Free translation by Tobias Micklitz)
Einstein 1910 (reversal of Boltzmann formula):

For any two independent systems $A$ and $B$,
the likelihood function should satisfy

$$W(A + B) = W(A) W(B) \quad \text{(Einstein principle)}$$

$q = 1$:

$$S_{BG} = k_B \ln W \quad \text{hence} \quad W \propto e^{S_{BG}/k_B} \quad \text{hence}$$

$$W(A+B) \propto e^{S_{BG}(A+B)/k_B} = e^{S_{BG}(A)/k_B+S_{BG}(B)/k_B} = e^{S_{BG}(A)/k_B} e^{S_{BG}(B)/k_B} \propto W(A) W(B) \text{ OK!}$$

$q = 1$:

$$S_{q} = k_B \ln_q W \quad \text{hence} \quad W \propto e^{S_{q}/k_B} \quad \text{hence}$$

$$W(A+B) \propto e^{S_{q}(A+B)/k_B} = e^{S_{q}(A)/k_B+S_{q}(B)/k_B+(1-q)[S_{q}(A)/k_B][S_{q}(B)/k_B]}$$

$$= e^{S_{q}(A)/k_B} e^{S_{q}(B)/k_B} \propto W(A) W(B) \text{ OK } \forall q!$$
On a $q$-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics

Sabir Umarov, Constantino Tsallis and Stanly Steinberg

Generalization of symmetric $\alpha$-stable Lévy distributions for $q>1$

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**CENTRAL LIMIT THEOREM**

\( N^{1/[\alpha(2-q)]} \) - scaled attractor \( F(x) \) when summing \( N \to \infty \) \( q \)-independent identical random variables

with symmetric distribution \( f(x) \) with \( \sigma_Q \equiv \int dx \ x^2 [f(x)]^Q / \int dx \ [f(x)]^Q \left( Q = 2q - 1, q_1 = \frac{1+q}{3-q} \right) \)

<table>
<thead>
<tr>
<th>( q = 1 ) [independent]</th>
<th>( q \neq 1 ) (i.e., ( Q = 2q - 1 \neq 1 )) [globally correlated]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_Q &lt; \infty ) (( \alpha = 2 ))</td>
<td>( F(x) = G_q(x) \equiv G_{(3q_1-1)/(1+q_1)}(x) ), with same ( \sigma_Q ) of ( f(x) )</td>
</tr>
<tr>
<td>( F(x) = G_q(x) \equiv G_{(3q_1-1)/(1+q_1)}(x) ), with same ( \sigma_1 ) of ( f(x) )</td>
<td>( G_q(x) \sim \begin{cases} G(x) &amp; \text{if }</td>
</tr>
<tr>
<td>( \text{Classic CLT} )</td>
<td>with ( \lim_{q \to 1} x_c(q,2) = \infty )</td>
</tr>
<tr>
<td>( \sigma_Q \to \infty ) (( 0 &lt; \alpha &lt; 2 ))</td>
<td>S. Umarov, C. T. and S. Steinberg, Milan J Math 76, 307 (2008)</td>
</tr>
<tr>
<td>( F(x) = L_{q,\alpha} ), with same (</td>
<td>x</td>
</tr>
<tr>
<td>( L_{q,\alpha} \sim \begin{cases} G(x) &amp; \text{if }</td>
<td>x</td>
</tr>
<tr>
<td>( \text{Levy-Gnedenko CLT} )</td>
<td>( \text{(intermediate regime)} )</td>
</tr>
<tr>
<td>S. Umarov, C. T., M. Gell-Mann and S. Steinberg</td>
<td>( \text{J Math Phys 51, 033502 (2010)} )</td>
</tr>
</tbody>
</table>
In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92) [the partition function, as we call it nowadays] to have a finite valued, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space [...]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. [...]. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).
TIME-EVOLVING STATISTICS OF CHAOTIC ORBITS OF CONSERVATIVE MAPS IN THE CONTEXT OF THE CENTRAL LIMIT THEOREM

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CONSERVATIVE MC MILLAN MAP:

\[ x_{n+1} = y_n \]

\[ y_{n+1} = -x_n + 2\mu \frac{y_n}{1 + y_n^2} + \varepsilon y_n \]

\[ \mu \neq 0 \iff \text{nonlinear dynamics} \]

\[(\mu, \varepsilon) = (1.6, 1.2) \quad \text{and} \quad (\lambda_{\text{max}} \approx 0.05)\]

\[N = 2^9\]  \quad  \[N = 2^{13}\]  \quad  \[N = 2^{16}\]

\[N = 2^{18}\]  \quad  \[2^{18} \leq N \leq 2^{19}\]  \quad  \[2^{19} \leq N \leq 2^{20}\]

FIG. 10. Structure of phase space plot of Mc. Millan perturbed map for parameter values \(\mu = 1.6\) and \(\varepsilon = 1.2\), starting form a randomly chosen initial condition in a square \((0, 10^{-6}) \times (0, 10^{-6})\), and for \(i = 1 \ldots N\) \((N = 2^{10}, 2^{13}, 2^{16}, 2^{18})\) iterates.

\[ p \propto e^{-\beta(z/\sigma)^2} \]

with \((q, \beta) = (1.6, 4.5)\)

CLASSICAL LONG-RANGE-INTERACTING MANY-BODY HAMILTONIAN SYSTEMS

\[ V(r) \sim -\frac{A}{r^\alpha} \quad (r \to \infty) \quad (A > 0, \quad \alpha \geq 0) \]

integrable if \( \alpha / d > 1 \) (short-ranged)

non-integrable if \( 0 \leq \alpha / d \leq 1 \) (long-ranged)
Influence of the interaction range on the thermostatistics of a classical many-body system

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