Disagreement, Speculation, and Aggregate Investment*

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Abstract

We study the effects of speculation caused by heterogeneous beliefs in a dynamic general equilibrium production economy. We characterize the impact of speculative production allocation risk on equilibrium quantities, asset prices, portfolios and financial trade. Speculation generates endogenous stochastic volatility of aggregate consumption growth, investment growth and equity returns. With low risk aversion, speculation increases stock return volatility and leads to procyclical investment-capital ratios, procyclical Tobin's q and countercyclical consumption-capital ratios. For all preferences we consider, interest rates are procyclical, the equity risk premium is countercyclical and stock return volatility can be either procyclical or countercyclical according to the level of the consumption share of optimists. The economy also features a large amount of financial leverage as investors speculate on their beliefs.

1 Introduction

How does financial speculation affect real investment and economic growth? Financial markets allow investors with differences of opinion to trade on their beliefs and equilibrium asset prices reflect investors beliefs' heterogeneity. Asset prices also affect firms' cost of capital—financial speculation affects a value maximizing firms' real investment decision through the cost of capital. We develop a simple production economy to study how speculation changes the allocation of capital in a dynamic general equilibrium economy. We use our model to study the impact of differences of opinion on the allocation of aggregate output between consumption and investment and the resulting speculative trade of financial assets. Financial speculation can lead to procyclical real investment rates, can increase real investment rate volatility and can lead to variation in Tobin's q. Financial speculation also causes interest rates to be procyclical and the equity premium to be countercyclical, and leads to endogenous stochastic volatility in aggregate consumption growth and equity returns.

Our dynamic general equilibrium economy features heterogeneous investors, complete financial markets and capital adjustment costs. Each investor observes the capital stock, output, aggregate consumption and aggregate investment. All investors agree on the capital adjustment costs but, for any investment rate, investors disagree on the expected capital growth and expected output growth rates. Accordingly, each investor in isolation would choose a different allocation of consumption and investment.

If investors agree on the drift of aggregate capital growth, then there is no financial speculation and the equilibrium investment-capital ratio is unaffected by productivity shocks. Under agreement, the conditional moments of asset prices including the riskless interest rate, the market price of risk, stock return volatility and the equity premium are constant and there is no financial trade. If investors disagree on the drift of aggregate capital growth, then there is financial speculation and the equilibrium investment capital ratio changes with productivity shocks. The conditional moments of asset prices including the riskless interest rate, the market price of risk, stock return volatility and the equity premium change with productivity shocks and there is financial trade. Since interest rates and the market price of risk are stochastic, the aggregate cost of capital is stochastic leading to variation in investment and stochastic volatility in aggregate consumption growth.

In our model all investors have the same preferences but they have differences of opinion. As a result, they trade financial assets guided by their beliefs so that, in equilibrium, positive productivity shocks increase the optimist's share of aggregate consumption and negative productivity shocks decrease the optimist's share of aggregate consumption. Such a consumption sharing rule is also common in endowment economies with differences of opinion, and is often called sentiment risk (Dumas et al. [2009]). Optimists insure pessimists against bad productivity shocks and receive a premium from the pessimists to bear that risk. In order to implement the consumption allocation, optimists hold levered positions in the stock market while pessimists hold bonds and, in some cases, even short the stock. A positive productivity shock increases the interest rate and decreases the market price of risk.

The effect of sentiment on the dynamics of aggregate investment and aggregate consumption is a new channel that appears exclusively in production economies in which investment and consumption are endogenous. Depending on the investors' common risk aversion, the cost of capital can either increase or decrease with positive productivity shocks, being either procyclical or countercyclical. If the investors' common risk aversion is higher than one, then the aggregate cost of capital and the consumption-capital ratio are both procyclical, while the investment-capital ratio and Tobin's q are both countercyclical. Aggregate consumption growth is more volatile than aggregate investment growth, and the stock return is less volatile than output growth. If the investors' common risk aversion is lower than one, then the aggregate cost of capital and the consumption-capital ratio are countercyclical, while the investment-capital ratio and Tobin's q are procyclical. Aggregate consumption growth is less volatile than aggregate investment growth, and the stock return is more volatile than output growth.

Differences of opinion lead to relationships between the investment-capital ratio and stock returns. Cochrane [1991] shows empirically that the investment-capital ratio negatively predicts stock returns. Such a relationship occurs in our model for both types of preferences we choose in good times, when the optimist's consumption share is sufficiently high. When the common risk aversion is lower than one we obtain, in addition, that the investment-capital ratio negatively predicts the equity premium too, but again only in good times, when the optimist's consumption share is sufficiently high.

Our work is related to the larger literature that studies the asset pricing implications of heterogenous beliefs in endowments economies. Basak [2005] shows how to characterize asset prices using the martingale approach in an endowment economy; Gallmeyer and Hollifield [2008] study the impact of a short-sales constraint with heterogeneous beliefs in an endowment economy; Kogan et al. [2006] show that heterogeneous beliefs can have a significant impact on asset prices even when irrational investors are a small fraction of the investors in the economy; David [2008] studies an endowment economy with heterogenous beliefs showing that heterogeneous beliefs can significantly increase the equity premium relative to a homogeneous beliefs economy with a low level of risk aversion; Dumas et al. [2009] show that sentiment risk can have significant impacts on investors' optimal portfolios and equilibrium asset prices; and Dumas et al. [2011] show that heterogenous beliefs can help explain several puzzles in international finance. Bhamra and Uppal [forthcoming] show how to solve in closed form for equilibrium with both heterogeneous beliefs and heterogeneous preferences in endowment economies. Their model includes external habits, which allows the possibility of a stationary outcome with the appropriate choice of parameters. Borovička [2013] solves for the equilibrium in a model with recursive preferences and heterogeneous beliefs and shows that a stationary equilibrium exists for the appropriate choice of parameters. All these papers study endowment economies rather than the production economy we study.

Detemple and Murthy [1994] study an economy with heterogeneous beliefs and endogenous output in which all investors have logarithmic utility, which gives fixed Tobin's q and fixed price-dividend ratios. As a consequence of the logarithmic utility, heterogenous beliefs mainly affect interest rates. We allow for non-logarithmic investors and capital adjustment costs, which allows for time variation in Tobin's q and price-dividend ratios. Panageas [2005] studies the effects of heterogeneous beliefs of the type studied by Scheinkman and Xiong [2003] on the relation between Tobin's q and real investment rates. He shows in an economy with risk-neutral investors facing short-sales constraints that q is related to real investment rates. In contrast, our model features risk-averse investors so that whether investment and Tobin's q are procyclical or countercyclical depends on the magnitude of investors' risk aversion.

Buss et al. [2013] study the effects of different regulations in a production economy with heterogeneous beliefs and also show that heterogeneous beliefs have a strong impact on production and asset prices. There are several differences in modeling approaches. Buss et al. [2013] consider a discrete-time, discrete-state economy with a finite horizon. We consider a continuous-time infinite horizon infinite horizon economy. Buss et al. [2013] show that Tobin taxes and short-sales constraints can actually increase stock return volatility and that leverage constraints can reduce stock return volatility. They also show that imposing a leverage constraint can increase economic growth. Our focus is not on regulation, but instead on how the asset prices, risk premia, investors' portfolios, consumption and investment behave in the presence of heterogeneous beliefs.

Altı and Tetlock [2013] estimate a structural model in which they solve for an individual firm's optimal investment decision in the presence of heterogeneous beliefs. They provide empirical evidence that investors have over-confident and trend following beliefs in a partial equilibrium model, in which there is no feedback from the heterogeneous beliefs to the dynamics of aggregate consumption or aggregate asset prices, nor to investors' optimal portfolio strategies. We focus on the feedback from the heterogeneous beliefs to individual and aggregate consumption, to aggregate investment, to aggregate asset prices, and to investors' optimal portfolio strategies.

2 The model

We study a one sector continuous-time production economy in which we allow for differences of opinion among investors. The economy is a one-sector version of Cox et al. [1985] with capital adjustment costs as in the model posed by Eberly and Wang [2011] extended to include the possibility that investors have different beliefs about capital growth.

The model is set in continuous time with an infinite horizon. Let K_t denote the representative firm's capital stock, I_t the aggregate investment rate, and Y_t the aggregate output rate. The representative firm has a constant-returns-to-scale production technology:

$$Y_t = AK_t, \tag{1}$$

with constant coefficient $A > 0$. Capital accumulation has the dynamics:

$$dK_t = \Phi_z\left(I_t, K_t\right) dt + \sigma K_t dW_t^z; \qquad K_0 > 0, \tag{2}$$

where $\sigma > 0$ is the volatility of capital growth and W_t^z is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}^t\}, \mathcal{P}^z)$ with the objective probability measure $\mathcal{P}^z$ governing empirical realizations of the process. The Brownian motion dW_t^z captures productivity shocks. The function $\Phi_z\left(I_t, K_t\right)$ measures the effectiveness of converting investment goods into installed capital under the objective probability measure. There are two types of investors a and b who may disagree about the drift of the capital stock in Equation (2) above.

As in the neoclassical investment literature (i.e. Hayashi [1982]), the firm's adjustment cost is homogeneous of degree one in I_t and K_t:

$$\Phi_z\left(I_t, K_t\right) = K_t \phi_z\left(i_t\right), \tag{3}$$

where $i_t \equiv \frac{I_t}{K_t}$ is the firm's investment capital ratio and $\phi_z\left(i_t\right)$ is an increasing and concave function. We use the quadratic adjustment cost function

$$\phi_z(i_t) = i_t - \frac{1}{2}\theta i_t^2 - \delta_z, \tag{4}$$

where $\theta > 0$ is the adjustment cost parameter. When $\theta = 0$, there are no adjustment costs, the expected growth rate of capital is $\phi(i_t) = i_t - \delta_z$ and the model is a one-sector Cox et al. [1985] economy. One interpretation of the parameter δ_z is the expected depreciation rate. Alternatively, the economy can be reformulated as an economy with deterministic dynamics for capital, and a stochastic productivity process. We report the details in the Appendix.

We denote by $c_t \equiv \frac{C_t}{K_t}$ the aggregate consumption-capital ratio. Using the investment-capital ratio and the AK production technology, the aggregate resource constraint is:

$$c_t + i_t = A. \tag{5}$$

The AK production technology has three useful properties. First, capital growth equals output growth: $\frac{dK_t}{K_t} = \frac{dY_t}{Y_t}$. Second, the investment-capital ratio is proportional to the investment-output ratio: $\frac{I_t}{Y_t} = \frac{1}{A}\frac{I_t}{K_t} = \frac{1}{A}i_t$. Third, the aggregate consumption-capital ratio is proportional to the aggregate consumption-output ratio: $\frac{C_t}{Y_t} = \frac{1}{A}\frac{C_t}{K_t} = \frac{1}{A}c_t$.

Investors have different beliefs about the drift of the capital stock for any given investment rate. Upon observation of K_t and i_t it is not possible for investors to distinguish whether shifts in capital are driven by productivity shocks or by the drift of capital growth. In order to capture different beliefs about the technology, investors can disagree about the value of δ_z.

There are two types of Investors, a and b. Type a investors are optimists because they believe that every unit invested transforms into installed capital at a rate higher than the rate believed by the pessimistic type b investors. Type a investors believe the parameter is δ_a and type b investors believe the parameters is $\delta_b > \delta_a$.[1] Since both types of investors

[1]We could let each investor learn about the true value of δ_z upon observation of realizations of the capital

have common information and not asymmetric information, they are aware of each others' different perception about δ_z, and each investor thinks investors of the other type are wrong. Hence, investors agree to disagree. The true value of the parameter δ_z lies between both investors' beliefs:

$$\delta_a < \delta_z < \delta_b. \tag{6}$$

We refer to type a investors as optimists and we refer to type b investors as pessimists.

We rewrite the dynamics of the capital stock in terms of investor specific Brownian motions. Define the Brownian motion process W_t^a on the optimist-specific filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}^t\}, \mathcal{P}^a)$ and define the Brownian motion process W_t^b on pessimist-specific filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}^t\}, \mathcal{P}^b\right)$. The dynamics of capital and output under each investors' beliefs are

$$\frac{dK_t}{K_t} = \frac{dY_t}{Y_t} = \phi_j\left(i_t\right)dt + \sigma dW_t^j; \qquad K_0 > 0, \tag{7}$$

where the subjective drift of capital growth is

$$\phi_j\left(i_t\right) = i_t - \frac{1}{2}\theta i_t^2 - \delta_j, \tag{8}$$

for $j = \{a, b, z\}$. The relation between the investor-specific Brownian motions is:

$$dW_t^b = \overline{\mu}dt + dW_t^a, \tag{9}$$

where

$$\overline{\mu} = \frac{\delta_b - \delta_a}{\sigma} > 0. \tag{10}$$

We call $\overline{\mu}$ disagreement.

Without loss of generality, we use the pessimist's probability measure as the reference

(or output) process. Instead, we use dogmatic beliefs to keep tractability of the model.

measure for our analysis.[2] From Equation (9) and Girsanov's theorem, the change from the pessimist's measure to the optimist's measure is given by the exponential martingale η_t with dynamics:[3]

$$\frac{d\eta_t}{\eta_t} = \overline{\mu} dW_t^b; \qquad \eta_0 = 1. \tag{11}$$

We compute expectations using the process η. For any $T > t$ measurable random variable

$$\mathbb{E}_t^a \left[X_T \right] = \mathbb{E}_t^b \left[\frac{\eta_T}{\eta_t} X_T \right], \tag{12}$$

where $\mathbb{E}_t^j$ denotes investor j's conditional expectations.

We call η_t sentiment following Dumas et al. [2009]. The role of η_t is to show how optimists over–estimate or under–estimate the probability of a state relative to pessimists. Optimists view positive productivity shocks as more probable than pessimists do, and hence optimists perceive higher expected output and capital growth rates than pessimists. Equations (7) and (11) completely characterize the evolution of the economy in the eyes of pessimists.

3 Equilibrium with agreement

Before presenting the results with disagreement, we summarize the model solution when all investors are of type j , so all of them agree on the value of the parameter δ_j, for $j = \{a, b, z\}$. The AK production technology and the quadratic adjustment cost imply that investment opportunities are constant so that the aggregate investment-capital ratio and Tobin's q are constant. We use such a simple benchmark to highlight the dynamic effects of disagreement on the economy.

All investors have power utility with the same risk aversion coefficient $1 - \alpha > 0$ and

[2] All the equilibrium quantities are the same if we use instead the optimist's probability measure as the reference measure.

[3] With Gaussian priors and Bayesian learning rather than dogmatic priors, the disagreement process would be deterministic and the process η_t would have a deterministic time-varying diffusion coefficient rather than a constant diffusion coefficient.

subjective discount rate ρ, with $0 < \rho < 1$. Assuming complete markets, a competitive equilibrium allocation is the solution to the planner's problem:

$$\sup_{c_t} \mathbb{E}_0^j \left[\int_0^\infty e^{-\rho t} \frac{1}{\alpha} \left(K_t c_t \right)^\alpha dt \right], \tag{13}$$

subject to the capital accumulation rule in Equation (7) and to the aggregate resource constraint in Equation (5).

The social planner's value function is:[4]

$$V_j \left(K_t \right) = \Lambda_j \frac{1}{\alpha} K_t^\alpha, \tag{14}$$

where Λ_j is a constant reported in the Appendix. The solution for the investment-capital ratio i_j is

$$i_j^* = \frac{A + \frac{1-\alpha}{\theta} - \sqrt{\left(A - \frac{1-\alpha}{\theta} \right)^2 + 2 \frac{2-\alpha}{\theta} \left(\rho - \alpha \left[\frac{A}{2-\alpha} - \left(\delta_j + \frac{1}{2} \left(1 - \alpha \right) \sigma^2 \right) \right] \right)}}{2 - \alpha}. \tag{15}$$

From the aggregate resource constraint in Equation (5), the aggregate consumption-capital ratio is also constant: $c_j^* = A - i_j^*$.

Because i_j^* and c_j^* are constant, the equilibrium capital stock follows a geometric Brownian motion. Since aggregate output, aggregate consumption, and aggregate investment are all proportional to the capital stock, they also follow geometric Brownian motions:

$$\frac{dK_t}{K_t} = \frac{dY_t}{Y_t} = \frac{dC_t}{C_t} = \frac{dI_t}{I_t} = \phi \left(i_j^* \right) dt + \sigma dW_t^j; \tag{16}$$
$$K_0 > 0; \;\; Y_0 = AK_0 > 0; \;\; C_0 = \left(A - i_j^* \right) K_0 > 0; \;\; I_0 = i_j^* K_0 > 0.$$

The dynamics of all aggregate quantities depend on the underlying technology and preferences through the investors' optimal investment-capital ratio decision in Equation (15). The

[4] Given power utility and the linearly homogenous capital accumulation process, the value function is homogenous of degree α in capital.

equilibrium interest rate, market price of risk and Tobin's q are all constant.

4 Equilibrium with disagreement

Following Basak [2005], we compute the competitive equilibrium in our complete market setting from the solution to a planner's problem. The Appendix shows that the planner's problem is a weighted average of the expected utility of each investor:

$$\sup_{c_{a,t}+c_{b,t}=K_t c_t} \left\{ \mathbb{E}_0^b \left[\int_0^\infty e^{-\rho t} \frac{1}{\alpha} c_{b,t}^\alpha dt \right] + \lambda \mathbb{E}_0^a \left[\int_0^\infty e^{-\rho t} \frac{1}{\alpha} c_{a,t}^\alpha dt \right] \right\}, \tag{17}$$

where λ is the initial weight of the planner on the optimist,[5] and subject to the capital accumulation rule in Equation (7) and to the aggregate resource constraint in Equation (5). Using the sentiment process η_t, the planner's problem under the pessimist's probability measure is

$$\sup_{c_{a,t}+c_{b,t}=K_t c_t} \left\{ \mathbb{E}_0^b \left[\int_0^\infty e^{-\rho t} \frac{1}{\alpha} c_{b,t}^\alpha dt \right] + \lambda \mathbb{E}_0^b \left[\int_0^\infty \frac{\eta_t}{\eta_0} e^{-\rho t} \frac{1}{\alpha} c_{a,t}^\alpha dt \right] \right\}. \tag{18}$$

The objective function is maximized subject to the aggregate resource constraint in Equation (5), the capital accumulation rule in Equation (7), and the sentiment dynamics in Equation (11). There are two dimensions of this optimization problem: The optimal individual consumption allocation rule among the two investors, and the optimal production allocation between investment and aggregate consumption. We consider first the individual optimal consumption allocation, given a level of aggregate consumption-capital ratio $c(\eta_t)$.

The Appendix shows that the optimal consumption sharing rule for the investors is:

$$c_{a,t} = \omega(\eta_t) c(\eta_t) K_t; \qquad c_{b,t} = [1 - \omega(\eta_t)]\, c(\eta_t) K_t, \tag{19}$$

[5]The initial weight depends on the relative initial endowments of the two representative investors.

where $c(\eta_t)K_t$ is aggregate consumption at t and ω_t is the consumption share of the optimist:

$$\omega_t = \omega(\eta_t) \equiv \frac{\left(\lambda \frac{\eta_t}{\eta_0}\right)^{\frac{1}{1-\alpha}}}{1 + \left(\lambda \frac{\eta_t}{\eta_0}\right)^{\frac{1}{1-\alpha}}}. \tag{20}$$

From equation (20) the optimist's consumption share, ω_t, is monotonically related to the change of measure η_t. We therefore express the equilibrium in terms of ω_t because of its convenient domain: $\omega_t \in (0, 1)$. Applying Ito's Lemma, the volatility of ω_t, $\sigma_\omega(\omega_t)$, is:

$$\sigma_\omega(\omega_t) = \frac{1}{1-\alpha}\omega_t(1-\omega_t)\overline{\mu}. \tag{21}$$

The conditional variance of ω_t is highest when $\omega = 0.5$, that is when both investors have an equal share of aggregate consumption. In addition, the conditional variance of ω_t is monotonically decreasing in the investors' risk aversion $1 - \alpha$.

From Equation (19), the optimist's consumption $c_{a,t}$ reacts to productivity shocks through two channels. The first channel is the optimist's consumption share, ω_t, which is positively correlated with productivity shocks. This channel operates in endowment economies with heterogeneous beliefs as well. The second channel is aggregate consumption, which is itself driven by the optimist's consumption share. The second channel is not present in endowment economies with heterogeneous beliefs and is a consequence of the impact of sentiment on output allocation in our production economy. We call the second channel speculative production allocation risk. In production economies the impact of sentiment on aggregate consumption adds a new dimension of risk to speculation: By speculating on their beliefs, investors place bets not only on their shares of aggregate consumption, but also on the aggregate consumption that is being shared.

We now turn to the solution of the optimal production allocation problem, obtaining the optimal investment-capital ratio $i_t(\omega_t)$ and aggregate consumption-capital ratio $c_t(\omega_t)$ as functions of ω_t. In order to characterize the production allocation, we substitute the optimal individual consumption allocation in Equation (19) into the planner's problem in Equation (18), subject to the aggregate resource constraint in Equation (5), the capital accumulation rule in Equation (7), and the sentiment dynamics in Equation (11). By the homogeneity of the problem, the value function is of the form

$$V\left(K_t,\omega_t\right)=\frac{1}{\left(1-\omega_t\right)^{1-\alpha}}H\left(\omega_t\right)\frac{1}{\alpha}K_t^{\alpha}, \tag{22}$$

where H is a function to be solved. The solution is a set of functions $i\left(\omega_t\right)$, $c\left(\omega_t\right)$ and $H\left(\omega_t\right)$ satisfying the first order condition for the optimal investment-capital ratio, the Hamilton-Jacobi-Bellman equation, the aggregate resource constraint and the boundary conditions. We report the first order condition, the Hamilton-Jacobi-Bellman equation and the resulting ordinary differential equation for H in the Appendix.

The boundary conditions for the function H are:

$$\lim_{\omega_t\to 0}H\left(\omega_t\right)=\frac{1}{\left(A-i_b^*\right)^{1-\alpha}\left(1-\theta i_b^*\right)};\qquad \lim_{\omega_t\to 1}H\left(\omega_t\right)=\frac{1}{\left(A-i_a^*\right)^{1-\alpha}\left(1-\theta i_a^*\right)}, \tag{23}$$

where i_a^* and i_b^* are the constant investment-capital ratios that are consistent with homogeneous beliefs economies populated only by optimists or only by pessimists, respectively.[6] The boundary conditions are such that when the consumption share of optimists tends to zero or one, the function H converges to the homogeneous beliefs solution for each type of investor. We numerically solve for the functions $i\left(\omega_t\right)$, $c\left(\omega_t\right)$ and $H\left(\omega_t\right)$.

Using the solution for $c\left(\omega_t\right)$, we apply Ito's Lemma to characterize the dynamics of aggregate consumption $C_t=c\left(\omega_t\right)K_t$. The dynamics of aggregate consumption under type j investor's beliefs is

$$\frac{dC_t}{C_t}=\mu_C^j(\omega_t)dt+\sigma_C(\omega_t)dW^j;\qquad C_0=c\left(\omega_0\right)K_0, \tag{24}$$

[6] These solutions are given in Equation (15).

where

$$\sigma_C(\omega_t) = \sigma + \frac{c'(\omega_t)}{c(\omega_t)} \sigma_\omega(\omega_t), \tag{25}$$

and $c'(\omega_t)$ is the derivative of the consumption-capital ratio with respect to ω_t.

From Equation (25), aggregate consumption growth volatility is stochastic, being driven not only by the constant diffusion of output growth σ, but also by the aggregate consumption-capital ratio policy and the diffusion of the consumption share, $\frac{c'(\omega_t)}{c(\omega_t)}\sigma_\omega(\omega_t)$. Positive productivity shocks not only affect capital K_t and output Y_t, but they also change the allocation of production to aggregate consumption $c(\omega_t)$. The endogenous allocation of investment and aggregate consumption is the new channel by which speculation impacts aggregate consumption risk.

The state price density under the reference measure of the pessimist, ξ_t^b, is obtained from the marginal period-utility of the planner with respect to aggregate consumption:

$$\xi_t^b = e^{-\rho t} \{(1-\omega_t) c(\omega_t) K_t\}^{\alpha-1}. \tag{26}$$

The state price density captures the fundamental risk in K_t and also incorporates a sentiment risk factor. As η_t fluctuates, the consumption share ω_t fluctuates, affecting the pricing of financial securities. As it was the case for individual consumption, the state price density ξ_t^b is affected by sentiment through two channels: individual consumption shares $(1-\omega_t)$, and the aggregate consumption-capital ratio $c(\omega_t)$. Accordingly, the pricing of all financial assets are affected by sentiment through the optimal production allocation embedded in the associated aggregate consumption-capital ratio policy $c(\omega_t)$.

Because there is only one Brownian motion under any investors' subjective probability measure, two linearly independent securities are required for a complete market to implement the Pareto optimal allocation. We assume that there is a locally riskless bond with endogenous rate of interest r_t and also a stock with endogenous value P_t that pays a dividend stream equal to aggregate consumption C_t. That makes two securities, one of them being instantaneously risky, meaning that we can implement the competitive equilibrium with these securities.[7]

The equilibrium price of equity, or total wealth of the economy, is the discounted sum of all future dividends:

$$P_t = \mathbb{E}_t^b \left[\int_t^{\infty} \frac{\xi_u^b}{\xi_t^b} C_u du \right]. \tag{27}$$

Tobin's q is defined as the ratio of the market value of the firm P_t and the book value of the firm K_t. Tobin's q satisfies

$$q(\omega_t) = \frac{1}{\phi'(i(\omega_t))}. \tag{28}$$

The capital stock increases by $\phi'(i)$ per marginal unit of investment, and each unit of capital is valued at q. The firm optimally chooses investment to equate $\phi'(i)\, q(i)$ to unity– the marginal cost of the investment. Because ϕ is concave in i as a consequence of adjustment costs, q is increasing in i. Using the quadratic specification for ϕ,

$$q(\omega_t) = \frac{1}{1 - \theta i(\omega_t)}, \tag{29}$$

and the price-dividend ratio is:

$$\frac{P_t}{C_t} = q(\omega_t) \div c(\omega_t) = \frac{1}{[1 - \theta i(\omega_t)]\,[A - i(\omega_t)]}. \tag{30}$$

In addition, the definition of Tobin's q implies that $P_t = q(\omega_t) K_t$. Accordingly, from Ito's lemma, the stock return volatility is

$$\sigma_P(\omega_t) = \sigma + \frac{q'(\omega_t)}{q(\omega_t)} \sigma_\omega(\omega_t), \tag{31}$$

and $q'(\omega_t)$ is the derivative of Tobin's q with respect to ω_t.

From Equation (31), stock return volatility is stochastic, being driven not only by the

[7] We require that the stock has non-zero diffusion in order to implement the equilibrium. We verify that stock volatility is non-zero in our numerical implementation.

constant diffusion of capital growth σ, but also by Tobin's q and the diffusion of the consumption share, $\frac{q'(\omega_t)}{q(\omega_t)}\sigma_\omega(\omega_t)$. Positive productivity shocks drive Tobin's q through the investment-capital ratio and sentiment.

Starting from the equilibrium pricing measure in Equation (26) and the aggregate resource constraint in Equation (5), we use Ito's lemma to obtain the interest rate and market prices of risk. The interest rate r_t is

$$\begin{aligned}
r(\omega_t) = \rho + (1-\alpha)\phi_b(A - c(\omega_t)) - \frac{1}{2}(1-\alpha)(2-\alpha)\sigma^2 \\
+ (1-\alpha)\omega_t\overline{\mu}\sigma - \frac{1}{2}\frac{\alpha}{1-\alpha}\omega_t(1-\omega_t)\overline{\mu}^2 \\
-(1-\omega_t)\frac{c'(\omega_t)}{c(\omega_t)}\Bigg\{(1-\alpha)\omega_t\overline{\mu}\sigma - \frac{1}{2}\frac{\alpha}{1-\alpha}\omega_t(1-\omega_t)\overline{\mu}^2 \\
\times\left[\frac{1}{\alpha}\omega_t\left((2-\alpha)\frac{c'(\omega_t)}{c(\omega_t)} - \frac{c''(\omega_t)}{c'(\omega_t)}\right) - \frac{1-2\omega_t}{1-\omega_t}\right]\Bigg\}. \qquad (32)
\end{aligned}$$

The interest rate has the familiar structure from equilibrium models of disagreement in endowment economies adjusted for our endogenous consumption process. The first term on the first line captures time preference, the second term on the first line is the standard wealth effect, and the third term on the first line is the standard precautionary saving effect. The second line includes terms driven by disagreement and speculation that are standard in endowment economies. The third and fourth lines incorporate additional terms that appear as a consequence of speculative production allocation risk. This risk generates endogenous variation in the aggregate consumption-capital ratio $c(\omega_t)$, as well as its first and second derivatives.

The market price of risk for the pessimist is:

$$\kappa_{b,t}(\omega_t) = (1-\alpha)\sigma_C(\omega_t) - \omega_t\overline{\mu}, \qquad (33)$$

and the market price of risk for the optimist is:

$$\kappa_{a,t}\left(\omega_t\right) = \left(1-\alpha\right)\sigma_C(\omega_t) + \left(1-\omega_t\right)\overline{\mu}. \tag{34}$$

Similar to endowment economies with heterogeneous beliefs, the first term represents the market price of aggregate consumption risk, and the second term represents the market price of sentiment risk. The main difference is that the market price of aggregate consumption risk is now driven by sentiment itself, because of the endogenous stochastic volatility of aggregate consumption growth, as shown in Equation (25). Importantly, the additional source of risk driven by sentiment influences the market price of risk relative to the endowment economy from the perspective of both investors by the same magnitude.

We compute the investors' portfolios in two steps. In the first step, we use the solution to the central planner's problem to obtain the process for each individual investor's optimal wealth process. In the second step, we compute the self-financing portfolio strategy that replicates the wealth processes. Details are given in the Appendix.

We solve for the optimal individual wealth of the pessimist and obtain the optimal individual wealth of the optimist from market clearing. We interpret the pessimist's wealth, $X_{b,t}$, as a security that pays a dividend at a rate equal to his individual consumption, $c_{b,t}$. The investors' wealths are

$$X_{b,t} = D_b\left(\omega_t\right)K_t \quad \text{and} \quad X_{a,t} = \left[q\left(\omega_t\right) - D_b\left(\omega_t\right)\right]K_t, \tag{35}$$

where $D_b\left(\omega_t\right)$ is the solution to an ordinary differential equation reported in the Appendix. The boundary conditions for the differential equation are such that when the consumption share of the optimist tends to zero, the pessimist's individual wealth converges to the value of equity in a homogeneous beliefs economy populated by pessimists only. When the consumption share of the optimist tends to one, the pessimist's wealth converges to zero. We solve numerically for the process $D_{b,t}$.

Following Cox and Huang [1989], the desired holding of equity for each investor can be calculated from the ratio of each investor's individual wealth diffusion and the stock price diffusion. The pessimist's holdings of equity shares is

$$\zeta_{P_t}^{b} = \frac{D_b(\omega_t)\,\sigma + D_b'(\omega_t)\,\sigma_\omega(\omega_t)}{q(\omega_t)\,\sigma + q'(\omega_t)\,\sigma_\omega(\omega_t)}, \tag{36}$$

and the pessimist's holdings of the locally riskless bond follow from the budget constraint:

$$\zeta_{B_t}^{b} = X_{b,t} - \zeta_{P_t}^{b} P_t. \tag{37}$$

5 Quantitative analysis

To illustrate the effect of disagreement on investment, consumption allocations, asset prices and portfolios, we present numerical examples. Our goal is not to match the magnitude of particular moments in the data, but we would like to work with reasonable parameter values. We specify parameter values based on the calibration in Eberly and Wang [2011] for an economy similar to ours but with agreement. Compared to them, we select higher values for the volatility of output growth $\sigma = 11\%$ and the adjustment costs $\theta = 15$, but choose a lower value for the subjective discount rate $\rho = 2.5\%$.[8] We do this in order to produce reasonable levels of stock return volatility, Tobin's q and interest rate, respectively. Our parameter values are given in Table 1.

The objective beliefs lie half-way between the pessimist and optimist beliefs, so neither the pessimist nor the optimist is more accurate than the other. Therefore, despite the fact that the model is non-stationary, as is typical in heterogeneous agents models', either investor type could dominate the economy in the long run with equal probability.[9]

[8]Eberly and Wang [2011] choose values $\sigma = 10\%$, $\theta = 10$ and $\rho = 4\%$, respectively.

[9]Examples of non-stationary models with heterogeneous agents include: Basak [2005], David [2008], Dumas et al. [2009], Longstaff and Wang [2012], Panageas [2005] Scheinkman and Xiong [2003], and Yan [2008]. Yan [2008] and Dumas et al. [2009] show that, in setups similar to ours, it takes a long time for any

5.1 Homogeneous beliefs

As a first exercise, we compare economies populated by investors with homogeneous beliefs but with different perceptions and different preferences, as described in Section 3. These will serve as boundary values when we study our general model of heterogeneous beliefs. Table 2 reports some basic statistics for the benchmark economies under agreement and according to objective (z), pessimist (b) or optimist (a) beliefs. In each economy, the beliefs are common among all investors, and we change the beliefs of the investors populating the economy in each column. In the first column, all investors have objective beliefs, in the second column all investors are pessimists and in the third column, all investors are optimists.

Panel A of the Table reports results where all investors have risk-aversion coefficient of 0.5; we refer to this economy as the low risk aversion economy. Panel B reports results where all investors have risk-aversion coefficients of 2.5; we refer to this economy as the high risk aversion economy. Note that with homogeneous beliefs the stock return volatility, market price of risk and equity premium do not depend on investors' perception about the mean depreciation, but only on risk aversion and output growth volatility. For that reason, the stock return volatility, market price of risk and equity premium do not change across columns in any of the two panels. Naturally, the economy with higher risk aversion generates higher stock return volatility, market price of risk and equity premium. In both panels, optimists perceive a higher expected output growth for any level of investment. For that reason, the usual wealth effect implies that the interest rate is relatively higher in economies populated by optimists and relatively lower in economies populated by pessimists. Because the equity premium is the same regardless of investor's perceptions, the interest rates drive expected stock returns, which are relatively higher for optimists and relatively lower for pessimists.

Turning to the low risk aversion economy in Panel A, optimists choose to allocate more output to investment and less to consumption. Accordingly, Tobin's q and the price-dividend

investor to dominate the economy in the long run. Borovička [2013] shows that a stationary distribution is possible with recursive preferences in an endowment economy and Bhamra and Uppal [forthcoming] show that a stationary distribution is possible in an endowment economy with external habits.

ratio are higher in an economy populated only by optimists. By contrast, the high risk aversion economy in Panel B shows that optimists choose to allocate less output to investment and more to consumption. Accordingly, Tobin's q and the price-dividend ratio are lower in an economy populated only by optimists.

5.2 The effects of heterogeneous beliefs

The optimist's consumption share, driven by sentiment, is the single state variable characterizing the equilibrium because capital stock affects some variables proportionally only. Accordingly, the dynamics of all the equilibrium prices, quantities and portfolios are driven by the dynamics of the optimist's consumption share, ω_t. Variables that are increasing in ω_t are procyclical because positive productivity shocks increase ω_t. The reason is that optimists have placed bets on good states, so they get to consume a higher consumption share when those states occur.

On figures 1 to 6, the x-axis is always the optimist's consumption share, ω_t. In all figures, we plot the case of low risk aversion on the left and the case with high risk aversion on the right. In figures that contain two lines, the solid line corresponds to our model and the dashed line corresponds to a benchmark economy populated by homogeneous investors that that agree on the true value δ_z. In figures that contain three lines, the solid line corresponds to the objective (z) measure, the dashed-dotted line corresponds to the optimist's (a) measure and the dashed line corresponds to the pessimist's (b) measure.

We start our analysis by examining the volatility of the optimist's consumption share, $\sigma_\omega(\omega_t)$, given in Equation (21). Figure 1 plots the volatility of the consumption share. In both economies $\sigma_\omega(\omega_t)$ is a quadratic function of ω_t, attaining its highest value at $\omega_t = 0.5$ and tends to zero when ω_t tends to zero or one. When both investors have a similar consumption share, shocks generate bigger swings in the consumption shares. Comparing across economies, the consumption share is more volatile at every level of the consumption share in the lower risk aversion economy than in the higher risk aversion economy.

5.2.1 Investment and consumption

Figure 2 plots the investment-capital ratio $i(\omega_t)$, the expected capital growth $\phi(i(\omega_t))$ and Tobin's q in both economies. The investment-capital ratio is procyclical in the low risk aversion economy but countercyclical in the high risk aversion economy. Clearly, the substitution effect is stronger in the case of low risk aversion and the wealth effect is stronger in the case of high risk aversion.[10] Our adjustment costs remain low enough that the investment-capital ratio drives expected capital growth (which equals expected output growth). Therefore, driven by $i(\omega_t)$, the expected capital growth $\phi(i(\omega_t))$ is procyclical in the low risk aversion economy but countercyclical in the high risk aversion economy. This is the case under all measures, but clearly optimists always perceive a higher expected capital growth than pessimists, while the expectation under the objective measure is in between. Finally, from the optimality condition in Equation (28), Tobin's q is increasing in the investment-capital ratio. Accordingly, Tobin's q is procyclical in the low risk aversion economy but countercyclical in the high risk aversion economy.

Figure 3 plots the capital-consumption ratio $c(\omega_t)$, the volatility of aggregate consumption growth and the volatility of aggregate investment growth for both economies. From the market clearing condition in Equation (5) it is clear that the consumption-capital is negatively correlated with the investment-capital ratio. Therefore, the consumption-capital ratio is countercyclical in the low risk aversion economy but procyclical in the high risk aversion economy. As it was the case for the investment-capital ratio, the substitution and wealth effects explain the difference among the two economies.

The second row of Figure 3 plots aggregate consumption growth volatility, $\sigma_C(\omega_t)$, given in Equation (25). In the low risk aversion economy, aggregate consumption growth volatility is lower than the volatility of capital growth and is a U-shaped function: it is countercyclical for low levels of the optimist's consumption share and procyclical for high levels of the opti-

[10]Given our preference choice, the low risk aversion economy can be understood as one with high elasticity of intertemporal substitution. Similarly the high risk aversion economy can be understood as one with low elasticity of intertemporal substitution.

mist's consumption share. In the high risk aversion economy, aggregate consumption growth volatility is higher than the volatility of capital growth and is an inverted U-shaped function: it is procyclical for low levels of the optimist's consumption share and countercyclical for high levels of the optimist's consumption share. These effects are clearly understood from inspection of Equation (25). The first row in Figure 3 shows that in the low risk aversion economy $\frac{c'(\omega_t)}{c(\omega_t)} < 0$ and in the high risk aversion economy $\frac{c'(\omega_t)}{c(\omega_t)} > 0$, that explains why the volatility of consumption growth is lower than the volatility of output growth with low risk aversion but higher than the volatility of output growth with high risk aversion. The shapes in the figures follow from the volatility of the optimist's consumption share, discussed in Figure 1. The last row in Figure 3 plots aggregate investment growth volatility which, due to market clearing, mirrors that of consumption growth volatility. We include both to highlight that, despite the fact that the volatility of output growth is constant, both the volatility of consumption growth and the volatility of investment growth are stochastic. Moreover, the volatility of consumption growth is lower than the volatility of investment growth in the low risk aversion economy, and the reverse is true in the high risk aversion economy.

5.2.2 Asset prices

The first row of Figure 4 plots stock return volatility, $\sigma_P(\omega_t)$, given in Equation (31). In the low risk aversion economy, stock return volatility is higher than the volatility of capital growth and is a U-shaped function: it is procyclical for low levels of the optimist's consumption share and countercyclical for high levels of the optimist's consumption share. In the high risk aversion economy, stock return volatility is lower than the volatility of capital growth and is an inverted U-shaped function: it is countercyclical for low levels of the optimist's consumption share and procyclical for high levels of the optimist's consumption share. These effects are clearly understood from inspection of Equation (31). The third row in Figure 2 shows that in the low risk aversion economy $\frac{q'(\omega_t)}{q(\omega_t)} > 0$ and in the high risk aversion economy $\frac{q'(\omega_t)}{q(\omega_t)} < 0$, that explains why the stock return volatility is higher than the volatility of output growth with low risk aversion but lower than the volatility of output growth with high risk aversion. The shapes in the figures follow from the volatility of the optimist's consumption share, discussed in Figure 1.

The middle row of Figure 4 plots the market price of risk in the heterogeneous beliefs economy under the objective, optimist's and pessimist's measures. The objective measure is the solid line, the optimist's measure is the dashed-dotted line and the pessimist's measure is the dashed line. The market prices of risk in our production economy have the same structure as in endowment economies. The key difference is that aggregate consumption risk $\sigma_{C,t}(\omega_t)$ now incorporates speculative production allocation risk driven by sentiment. The higher the proportion of pessimists, the bigger the equity premium premium they are willing to pay in exchange for insurance against negative productivity shocks.

The final row of Figure 4 plots the equity premium in the heterogeneous beliefs economy under the objective, optimist's and pessimist's measures. The objective measure is the solid line, the optimist's measure is the dashed-dotted line and the pessimist's measure is the dashed line. The equity premium is just the product of the stock return volatility and the market price of risk, so they follow from the previous objects analyzed. For low levels of ω_t the steep slope of stock return volatility generates a small hump in the equity premium. Otherwise, the equity premium is decreasing in ω_t driven mostly by the effect of the market price of risk.

To further illustrate the impact the endogenous investment and consumption on the equity premium we study our model's implied CAPM. Instead of the standard consumption CAPM, we disentangle aggregate consumption risk into fundamental productivity risk and sentiment risk, including speculative production allocation risk driven by sentiment. For any asset S_t the following CAPM holds for the pessimist:

$$\mu_S^b - r = (1-\alpha)\left\{Cov\left(\frac{dS_t}{S_t}, \frac{dY_t}{Y_t}\right) + \left[-\frac{\omega_t}{1-\omega_t} + \frac{c'(\omega_t)}{c(\omega_t)}\right] Cov\left(\frac{dS_t}{S_t}, \frac{d\omega_t}{\omega_t}\right)\right\}, \tag{38}$$

with a similar pricing formula holding for the optimist.

The first term is the standard aggregate risk factor $\frac{dK_t}{K_t} = \frac{dY_t}{Y_t}$, any risky security's risk premium is positively related to the covariance of its return with the change in the fundamental risk factor equal to aggregate output growth. In endowment economies this term is aggregate consumption risk $\frac{dY_t}{Y_t} = \frac{dC_t}{C_t}$, as a consequence of the fixed aggregate consumption-capital ratio. We focus on the expression in square brackets multiplying the sentiment risk factor.

We interpret the first term in the squared bracket as follows. For the pessimist, an increase in ω_t is unfavorable because his consumption share decreases as a consequence of losses from speculation. Accordingly, he would be willing to pay an insurance (negative) premium for assets positively correlated with $\frac{d\omega_t}{\omega_t}$. For the optimist, an increase in ω_t is favorable because his consumption share increases as a consequence of gains from speculation. Hence, he would require a (positive) risk premium for assets positively correlated with $\frac{d\omega_t}{\omega_t}$.

The last term in the squared bracket is new. It does not appear in endowment economy models with heterogeneous beliefs, and stems from the pricing of sentiment driven by speculative production allocation risk. The sign of that effect depends on the level of risk aversion. In the low risk aversion economy, this term is negative. In this case, both investors require a smaller additional risk premium for any asset positively correlated with $\frac{d\omega_t}{\omega_t}$ because in states in which ω_t increases, aggregate consumption decreases for every unit of capital. In high risk aversion economy, this term is positive. In this case, both investors require a larger risk premium for any asset positively correlated with $\frac{d\omega_t}{\omega_t}$ because in states in which ω_t increases, aggregate consumption decreases for every unit of capital. For any level of risk aversion and as long as the risk of capital is already priced in the first component in the CAPM $\left(\frac{dK_t}{K_t} = \frac{dY_t}{Y_t}\right)$, the last term incorporates the risk of changes in aggregate consumption for a given level of output that are driven by sentiment only. Our model of disagreement in a production economy identifies additional terms in the standard CAPM that account for the risk of disagreement and speculation of beliefs through the impact on the aggregate consumption-investment decision.

The top row of Figure 5 plots the price-dividend ratio in the heterogeneous beliefs economy with a solid line and the price-dividend ratio in the homogeneous beliefs economy with the dashed line. From Equation (30), the price-dividend ratio is Tobin's q divided by the consumption capital ratio. In the low risk aversion economy, Tobin's q is monotonically increasing in the optimists' consumption share, the consumption-capital ratio is decreasing in the optimists' consumption share and the price-dividend ratio is increasing in the optimists' consumption share. In the high risk aversion economy, Tobin's q is monotonically decreasing in the optimists' consumption share, the consumption-capital ratio is decreasing in the optimists' consumption share and the price-dividend ratio is decreasing in the optimists' consumption share. Overall, the price-dividend ratio is procyclical in the low risk aversion economy and countercyclical in the high risk aversion economy.

The second row of Figure 5 plots the interest rate against the optimist's consumption share. The solid line is the interest rate in the economy with disagreement and the dashed line is the interest rate in an economy where all investors agree. In both the low risk aversion economy and the high risk aversion economy, interest rates are procyclical. The result follows because in good times the optimist's consumption share increases, raising the expected aggregate consumption growth for the average investor. The slope of the relationship between interest rates and the optimist's consumption share is flatter in the lower risk aversion economy, implying that interest rates are less sensitive to expected consumption growth with a lower risk aversion than with a higher risk aversion.

The third row of Figure 5 plots the expected stock return under the objective, optimist's and pessimist's measures. The objective measure is the solid line, the optimist's measure is the dashed-dotted line and the pessimist's measure is the dashed line. The expected return is obtained from adding the interest rate to the equity premium, both of which have been analyzed previously. In the low risk aversion economy the equity premium effect dominates and therefore the expected return is countercyclical. In the high risk aversion economy the interest rate effect dominates and therefore the expected return is procyclical. As it was the case with the equity premium and market price of risk, for any level of the optimist's consumption share, the expected return is higher for the optimist than for the pessimist, while the expected return under the objective measure lies in between.

5.2.3 Portfolios and leverage

Investors speculate on their beliefs by trading in the financial market. The pessimist lends to the optimist by purchasing the locally riskless bond. As a result, the pessimist switches his stock holdings into locally riskless debt, thereby decreasing the risk in his wealth. The optimist takes the funds obtained from selling the locally riskless bond to the pessimist and used the funds to purchase the stock.

The top row of Figure 6 plots the optimist's portfolio weight in equity and the middle row of the Figure plots the pessimist's portfolio weight in the stock in the heterogeneous beliefs economy. As a point of comparison, all investors' portfolio weights in the homogeneous beliefs economy are always one and leverage is zero. The portfolio weights equal the value of the investors' stock holdings divided by the investors' wealth. The optimist's wealth converges to the total value of equity and the optimist's equity holdings converge to all the equity as the consumption share goes to one, and the optimist's wealth and equity holdings both converge zero as the consumption share goes to zero. Similar results hold for the pessimist as the optimist's consumption share goes to zero.

In the low risk aversion economy, both the optimist's and pessimist's portfolio weight are always countercyclical. The pessimist shorts the stock and invests in bonds unless the optimist's consumption share is lower than 0.2. In the high risk aversion economy, the optimist's portfolio weight is procyclical for low levels of the optimist's consumption share and countercyclical for high levels of the optimist's consumption share, while the pessimist's portfolio weight is always countercyclical. Even though the pessimist's portfolio weight is countercyclical in both economies, the pessimist shorts the stock only in the low risk aversion economy. This difference is explained by the equity premium plotted in Figure 4—the premium becomes negative for a large enough consumption share in the low risk aversion economy and is always positive in the high risk aversion economy.

In both economies, the optimist tends to have extreme portfolio weights of the stock in bad times, financed by a large amount of leverage. Conversely, the pessimist has extreme portfolio weights of the bond in bad times.

In order to assess the importance of the credit market, we follow Longstaff and Wang [2012] and define the market-leverage ratio as the ratio of aggregate credit in the market to the total value of assets held by investors $\frac{|\zeta^j_{B_t}|}{P_t}$. The plots in the bottom panel panel of Figure 6 that aggregate leverage has an inverse U shape. The magnitudes of leverage show that there is plenty of credit market activity. Buss et al. [2013] show that reducing these amounts of leverage by means of a leverage constraint is an important tool to reduce volatility in a production economy with speculation caused by heterogeneous beliefs.

5.2.4 Tobin's q and equity returns

In the previous section, we reported the implications of disagreement for investment, aggregate consumption, the market price of risk, the interest rate, asset prices and portfolios in our production economy framework. Here, we explore the relationship between returns and Tobin's q. The CAPM derived in Equation (38), suggests that the relationship between Tobin's q or investment rates and returns is related to the sentiment risk that is priced in the CAPM.

Figure 7 plots equity premia and expected stock returns against Tobin's q. In each plot there are three lines. The solid line plots is under the objective measure z, the dotted-dashed line is under the optimists' measure, and the dashed line is under the pessimists' measure. In all cases, the equity premia and expected stock returns are highest under the optimists' measure, followed by the expectations under the objective measure z, and the lowest are the expectations under the pessimists' measure. This is intuitive, and we include equity premia and expected returns under all these three measures to highlight that the sign of the slopes is the same under all measures.

In the low risk aversion economy, both the equity premium and the expected stock return have an inverted U shape when plotted against the investment-capital ratio. To understand these results, we analyze the way in which the optimist's consumption share drives jointly the investment-capital ratio, the equity premium and the expected stock return. From Figure 2 we note that, in the low risk aversion economy the investment-capital ratio is monotonically increasing. In addition, figures 4 and 5 show that both the equity premium and expected stock return are increasing in very bad times but otherwise increasing. This explains why in Figure 7 the left plots show an inverted U shape for the equity premium and the expected stock return when plotted against the investment-capital ratio.

In the high risk aversion economy, the equity premium is increasing in the investment-capital ratio and the expected stock return is decreasing in the investment-capital ratio. To understand these results, we analyze the way in which the optimist's consumption share drives jointly the investment-capital ratio, the equity premium and the expected stock return. From Figure 2 we note that, in the high risk aversion economy the investment-capital ratio is monotonically decreasing. In addition, figure 4 shows that the equity premium is monotonically decreasing and 5 shows the expected stock return are monotonically increasing in very bad times but otherwise increasing. This explains why in Figure 7 the right plots show a positive relation between the equity premium and the investment-capital ratio, and a negative relation between the expected stock return and the investment-capital ratio.

Overall we obtain that the investment-capital ratio negatively predicts stock returns in good times, as shown empirically by Cochrane [1991]. When the common risk aversion is lower than one we obtain, in addition, that the investment-capital ratio negatively predicts the equity premium too, but again only in good times, when the optimist's consumption share is sufficiently high.

6 Conclusion

We provide a tractable continuous-time production framework to study the implications of disagreement on the allocation of aggregate investment and aggregate consumption, as well as on equilibrium asset prices, portfolios and financial trade. In production economies in which investment is chosen optimally, a new dimension of risk driven by disagreement and speculation emerges. The main intuition for this additional risk dimension is that disagreement affects not only the shares of aggregate consumption among investors with different views, but also the aggregate consumption to be shared among those investors.

We provide a characterization of the impact of speculative production allocation risk on the equilibrium allocation of output into consumption and investment, asset prices, portfolios and financial trade. We model the simplest possible type of sentiment risk: all the investors are dogmatic and have a fixed symmetric bias in their beliefs. As a consequence, disagreement risk has simple dynamics. Even in such a simple setting, our model produces endogenous stochastic volatility for aggregate consumption and several interesting implications for the equilibrium allocation of output into consumption and investment, asset prices, portfolios and financial trade. In an economy with low risk aversion, speculation leads to procyclical investment rates and Tobin's q, with countercyclical consumption rates. With the same preferences, speculation increases stock return volatility, which can be either procyclical or countercyclical according to the level of the optimist's consumption share. For all preferences we consider, the interest rates is procyclical and the equity risk premium is generally countercyclical. The economy also features a large amount of financial leverage, which is used by optimists to speculate on their beliefs.

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A Appendix

A.1 Alternative model with deterministic capital and stochastic productivity

In our model, capital dynamics are given in equations (7) and (8). Because A is a constant, output is proportional to capital and the dynamics of output growth mirror those of capital growth:

$$\frac{dY_t}{Y_t} = \left(i_t - \frac{1}{2}\theta i_t^2 - \delta_j\right) dt + \sigma dW_t^j; \qquad Y_0 = AK_0 > 0, \tag{1}$$

where $-\delta_j$ is the mean output productivity not driven by investment.

Alternatively, consider the following specification with deterministic dynamics for capital

$$\frac{d\widetilde{K}_t}{\widetilde{K}_t} = \left(i_t - \frac{1}{2}\theta i_t^2 - \delta\right) dt; \qquad \widetilde{K}_0 > 0 \tag{2}$$

but with stochastic productivity

$$\frac{d\widetilde{A}_t}{\widetilde{A}_t} = \psi_j dt + \sigma dW_t^j; \qquad \widetilde{A}_0 > 0. \tag{3}$$

It then follows from Ito's lemma that

$$\frac{d\widetilde{Y}_t}{\widetilde{Y}_t} = \left(i_t - \frac{1}{2}\theta i_t^2 + \mu_j\right) dt + \sigma dW_t^j; \qquad Y_0 = \widetilde{A}_0\widetilde{K}_0 > 0, \tag{4}$$

where $\mu_j = \psi_j - \delta$ is the mean net output productivity not driven by investment.

In the CIR specification in the main body of the text, output is proportional to capital, meaning that the investment-capital ratio is proportional to the investment-output ratio. We can therefore reinterpret the drift of output in Equation (4) as function of the investment-output ratio rather than the output-capital ratio. It is then clear that the two ways of modeling serve our objective: both of them involve disagreement of investors about output growth driven by disagreement about the mean productivity, whether it is defined by $-\delta_j$ in our CIR specification with stochastic capital and deterministic productivity, or by μ_j in the real business cycles specification with deterministic capital and stochastic productivity.

A.2 Solution with agreement

Given the conjectured value function in Equation (14), the solution is a pair of constants (i, Λ) that satisfy the first order condition

$$0 = A - i - \left[(1 - \theta i)\,\Lambda\right]^{\frac{1}{\alpha-1}} \tag{5}$$

and the Hamilton-Jacobi-Bellman equation

$$0 = (1-\theta i)^{\frac{\alpha}{\alpha-1}}\,\Lambda^{\frac{1}{\alpha-1}} + \alpha\phi(i) - \frac{1}{2}\alpha(1-\alpha)\sigma^2 - \rho. \tag{6}$$

The solution for Λ is

$$\Lambda_j^* = \frac{1}{\left(A - i_j^*\right)^{1-\alpha}\left(1-\theta i_j^*\right)} \tag{7}$$

and Equation (15) is the optimal investment rate.

A.3 Social planner's value function with disagreement

Let $V(\cdot,\cdot)$ be the value function of the planner in an economy with heterogeneous beliefs, where the first argument is capital stock, and the second argument captures the optimist's Pareto-weight. Also, let $V_a(K_0)$ be the expected utility of the optimist when the current aggregate capital stock is K_0 with $V_b(K_0)$ defined similarly. The planner's value function at time zero, with initial capital stock K_0 and weight λ on the optimist, satisfies the recursion:

$$\begin{aligned} V(K_0,\lambda) &= V_b(K_0) + \lambda V_a(K_0) \qquad (8)\\ &= \sup_{c_{a,t}+c_{b,t}=K_t c_t} \left\{ \mathbb{E}_0^b\left[\int_0^\infty e^{-\rho t}\frac{1}{\alpha}c_{b,t}^\alpha dt\right] + \lambda \mathbb{E}_0^a\left[\int_0^\infty e^{-\rho t}\frac{1}{\alpha}c_{a,t}^\alpha dt\right]\right\}\\ &= \sup_{c_{a,t}+c_{b,t}=K_t c_t} \mathbb{E}_0^b\left[\int_0^\infty e^{-\rho t}\left(\frac{1}{\alpha}c_{b,t}^\alpha + \lambda\frac{\eta_t}{\eta_0}\frac{1}{\alpha}c_{a,t}^\alpha\right)dt\right]\\ &= \sup_{c_{a,t}+c_{b,t}=K_t c_t} \mathbb{E}_0^b\left[\int_0^\tau e^{-\rho t}\left(\frac{1}{\alpha}c_{b,t}^\alpha + \lambda\frac{\eta_t}{\eta_0}\frac{1}{\alpha}c_{a,t}^\alpha\right)dt + \int_\tau^\infty e^{-\rho t}\left(\frac{1}{\alpha}c_{b,t}^\alpha + \lambda\frac{\eta_\tau}{\eta_0}\frac{\eta_t}{\eta_\tau}\frac{1}{\alpha}c_{a,t}^\alpha\right)dt\right]\\ &= \sup_{c_{a,t}+c_{b,t}=K_t c_t} \left\{ \mathbb{E}_0^b\left[\int_0^\tau e^{-\rho t}\left(\frac{1}{\alpha}c_{b,t}^\alpha + \lambda\frac{\eta_t}{\eta_0}\frac{1}{\alpha}c_{a,t}^\alpha\right)dt\right] + e^{-\rho\tau}\mathbb{E}_0^b\left[V\left(K_\tau, \lambda\frac{\eta_\tau}{\eta_0}\right)\right]\right\}, \end{aligned}$$

where the final line follows from the law of iterated expectations and the definition of the value function.

It then follows that the value function evaluated at time t with weight $\lambda\frac{\eta_t}{\eta_0}$ on the optimist satisfies

$$\begin{aligned} V\left(K_t, \lambda\frac{\eta_t}{\eta_0}\right) &= \sup_{c_{a,t}+c_{b,t}=K_t c_t} \mathbb{E}_t^b\left[\int_t^\infty e^{-\rho(s-t)}\left(\frac{1}{\alpha}c_{b,s}^\alpha + \lambda\frac{\eta_t}{\eta_0}\frac{\eta_s}{\eta_t}\frac{1}{\alpha}c_{a,s}^\alpha\right)ds\right]\\ &= \sup_{c_{a,t}+c_{b,t}=K_t c_t} \left\{ \mathbb{E}_0^b\left[\int_t^\infty e^{-\rho(s-t)}\frac{1}{\alpha}c_{b,t}^\alpha dt\right] + \lambda\frac{\eta_t}{\eta_0}\mathbb{E}_t^a\left[\int_t^\infty e^{-\rho(s-t)}\frac{1}{\alpha}c_{a,s}^\alpha ds\right]\right\}\\ &= V_b(K_t) + \lambda\frac{\eta_t}{\eta_0}V_a(K_t). \qquad (9) \end{aligned}$$

Accordingly, we obtain the boundary conditions for the value function

$$\lim_{\eta_t \to 0} V\left(K_t, \lambda \frac{\eta_t}{\eta_0}\right) = V_b(K_t); \qquad \lim_{\eta_t \to \infty} \frac{V\left(K_t, \lambda \frac{\eta_t}{\eta_0}\right)}{\lambda \frac{\eta_t}{\eta_0}} = V_a(K_t). \tag{10}$$

Using the definition of the consumption share in Equation (20) and the form of the value function in Equation (22), we obtain the boundary conditions in Equation (23).

Using market clearing in consumption: $c_{a,t} + c_{b,t} = K_t c_t$, it follows that

$$V\left(K_t, \lambda \frac{\eta_t}{\eta_0}\right) = V_b(K_t) + \lambda \frac{\eta_t}{\eta_0} V_a(K_t) \tag{11}$$

$$= \sup_{c_t} \mathbb{E}_0^b \left[\int_0^\infty e^{-\rho t} \frac{1}{\alpha} K_t^\alpha c_t^\alpha \left(1 + \left(\lambda \frac{\eta_t}{\eta_0}\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} dt \right] \tag{12}$$

and the planner's equilibrium can be decentralized using the consumption share in Equations (19) and (20).

A.4 Solution of the social planner's problem with disagreement

The Hamilton-Jacobi-Bellman (HJB) equation for the planner's problem is

$$\rho V = \sup_{i_t} \left\{ \frac{1}{\alpha} K_t^\alpha \left(A - i_t\right)^\alpha \left(1 + \left(\lambda \frac{\eta_t}{\eta_0}\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} + V_K \phi_b\left(i_t\right) K_t + \frac{1}{2} V_{KK} \sigma^2 K_t^2 \right.$$
$$\left. + \frac{1}{2} V_{\eta\eta} \eta_t^2 \overline{\mu}^2 + V_{\eta K} \eta_t K_t \overline{\mu} \sigma \right\}. \tag{13}$$

Using these dynamics of ω_t and our conjectured value function in Equation (22) we obtain the first order conditions:

$$0 = A - i\left(\omega_t\right) - \left(H\left(\omega_t\right)\left[1 - \theta i\left(\omega_t\right)\right]\right)^{\frac{1}{\alpha - 1}}. \tag{14}$$

Plugging these first order conditions back into the HJB we obtain the following ODE for $H\left(\omega_t\right)$:

$$0 = \left[1 - \theta i\left(\omega_t\right)\right]^{\frac{\alpha}{\alpha-1}} H\left(\omega_t\right)^{\frac{1}{\alpha-1}} + \alpha \phi_b\left(i\left(\omega_t\right)\right) - \frac{1}{2}\alpha\left(1-\alpha\right)\sigma^2 \tag{15}$$
$$+ \frac{1}{2}\overline{\mu}^2 \left(\frac{1}{1-\alpha}\right)^2 \omega_t\left(1-\omega_t\right)\left[\alpha\left(1-2\omega_t\right)\frac{H'\left(\omega_t\right)}{H\left(\omega_t\right)} + \omega_t\left(1-\omega_t\right)\frac{H''\left(\omega_t\right)}{H\left(\omega_t\right)} + \alpha\right]$$
$$+ \alpha\overline{\mu}\sigma\left[\frac{1}{1-\alpha}\omega_t\left(1-\omega_t\right)\frac{H'\left(\omega_t\right)}{H\left(\omega_t\right)} + \omega_t\right] - \rho.$$

A.5 Portfolios

We first show how to obtain equilibrium wealth and we then proceed to obtain portfolios.

Let $X_{b,t} = D_b(\omega_t) K_t$ be the wealth of the pessimist. Because $X_{b,t}$ is equivalent to a security that pays out a dividend equal to the optimal consumption of the pessimist, $c_{b,t}$, no arbitrage implies that $X_{b,t}$ satisfies

$$\mathbb{E}_t^b\left[d\left(\xi_t^b X_{b,t}\right)\right] + \xi_t^b c_{b,t} dt = 0. \tag{16}$$

Ito's lemma implies that the dynamics of $X_{b,t}$ are

$$\frac{dX_{b,t}}{X_{b,t}} = m(\omega_t)\, dt + n(\omega_t)\, dW_t, \tag{17}$$

where

$$m(\omega_t) = \phi_b(i(\omega_t)) + \frac{D_b'(\omega_t)}{D_b(\omega_t)}\sigma_\omega(\omega_t)\left[-\frac{1}{2}\frac{1}{1-\alpha}(2\omega_t - \alpha)\overline{\mu} + \sigma\right] + \frac{1}{2}\frac{D_b''(\omega_t)}{D_b(\omega_t)}\sigma_\omega^2(\omega_t),$$

$$n(\omega_t) = \sigma + \frac{D_b'(\omega_t)}{D_b(\omega_t)}\sigma_\omega(\omega_t) \tag{18}$$

and we have used the dynamics of K_t in Equation (7) and the dynamics of ω_t. In addition, we know that the state price density dynamics are given by:

$$\frac{d\xi_t^b}{\xi_t^b} = -r_t dt - \kappa_{b,t} dW_t. \tag{19}$$

Using Ito's lemma one more time we obtain the dynamics of $\xi_t^b X_{b,t}$, in terms of the dynamics of ξ_t^b and $X_{b,t}$ given in Equations (19) and (17), respectively. In particular, we get

$$\frac{d\left(\xi_t^b X_{b,t}\right)}{\left(\xi_t^b X_{b,t}\right)} = \frac{d\xi_t^b}{\xi_t^b} + \frac{dX_{b,t}}{X_{b,t}} + \frac{d\xi_t^b}{\xi_t^b}\frac{dX_{b,t}}{X_{b,t}}. \tag{20}$$

Plugging these dynamics into our no arbitrage restriction in Equation (16) and simplifying we get the differential equation for $D_b(\omega_t)$:

$$\begin{aligned} 0 =& \left[-r(\omega_t) + \phi_b(i(\omega_t)) - \kappa_b(\omega_t)\sigma\right] D_b(\omega_t) \\ &+ \frac{1}{1-\alpha}\omega_t(1-\omega_t)\overline{\mu}\left[-\frac{1}{2}\frac{1}{1-\alpha}(2\omega_t-\alpha)\overline{\mu} + \sigma - \kappa_b(\omega_t)\right] D_b'(\omega_t) \\ &+ \frac{1}{2}\left[\frac{1}{1-\alpha}\omega_t(1-\omega_t)\overline{\mu}\right]^2 D_b''(\omega_t) + (1-\omega_t)\left[A - i(\omega_t)\right], \end{aligned} \tag{21}$$

with boundary conditions

$$\lim_{\omega_t \to 0} D_b(\omega_t) = \frac{1}{\left(c_b^*\right)^{1-\alpha}\left(1-\theta i_b^*\right)}; \qquad \lim_{\omega_t \to 1} D_b(\omega_t) = 0. \tag{22}$$

The boundary conditions are intuitive and explained in are that the wealths of the investors converge to the single-agent wealths at the boundaries.

The diffusion coefficients are obtained by applications of Ito's lemma to the expressions for the wealth of the pessimist and the equity price, P_t, respectively. The portfolio construction in terms of diffusions follows from Cox and Huang (1989).

A.6 Implementing the planner's investment decisions

In equilibrium both investors agree on the firm's value, which is equal to the observed stock price. Therefore, the investment allocation can be implemented through a representative firm that chooses the investment plan to maximize firm value.

Lemma 1 *Given their subjective beliefs and taking prices as given, both investors agree on the investment plan that maximizes firm value.*

Proof. The optimal value maximizing plan for the firm from the perspective of Investor $j = \{a, b, z\}$ solves

$$\sup_i \mathbb{E}_t^j \left[\int_t^\infty \frac{\xi_s^j}{\xi_t^j} K_s \left(A - i_s\right) ds \right], \tag{23}$$

$$\text{s.t. } \frac{dK_t}{K_t} = \phi_j \left(i\left(\omega_t\right)\right) dt + \sigma dW_t^j, \tag{24}$$

$$\frac{d\xi_t^j}{\xi_t^j} = -r_t dt - \kappa_{j,t} dW_t^j, \tag{25}$$

where ξ_t^j is Investor j's state price density.

The maximand in the first equation is the observed stock price P_t. Because investors must agree on observed prices, the maximand is identical for any investor j. Equation (24) are the dynamics of capital under the subjective measure of investor j, and Equation (25) are the dynamics of the subjective state price densities.

Investor-specific state price densities ξ_t^j and sentiment η_t are related by:

$$\eta_t = \lambda \frac{\xi_t^b}{\xi_t^a}. \tag{26}$$

Using this relation, and the dynamics of sentiment in Equation (11), leads to consistency on the dynamics of the state price density ξ_t^j under the measure of any investor j in Equation (25).

Because the optimization problem is the same under the eyes of each investor j, both investors must agree on the optimal investment policy that maximizes firm value, given their subjective beliefs and assuming they take equilibrium prices as given. ■

	Parameter	Value
Expected depreciation (%)	δ	1.38
Volatility of output growth (%)	σ	11.00
Productivity	A	0.10
Time Preference (%)	ρ	2.50
Adjustment cost	θ	15.00
Disagreement	$\overline{\mu}$	0.25
Optimist Beliefs (%)	δ_a	0.00
Pessimist Beliefs (%)	δ_b	2.75

Table 1: **Parameter values:** The Table reports the common parameter values used in our numerical examples.

	Beliefs		
	Objective (δ)	Pessimistic (δ_b)	Optimistic (δ_a)
Panel A, High IES and Low Risk Aversion: $1-\alpha=0.5$			
Stock return volatility (%)	11.00	11.00	11.00
Market price of risk	0.055	0.055	0.055
Equity premium (%)	0.60	0.60	0.60
Expected output growth(%)	1.59	-0.03	3.17
Interest rate (%)	2.84	2.03	3.63
Expected stock return (%)	3.44	2.63	4.24
Investment-capital ratio (%)	4.44	3.80	5.19
Tobin's q	2.99	2.32	4.51
Price-dividend ratio	53.81	37.48	93.77
Panel B Low IES and High Risk Aversion: $1-\alpha=2.5$			
Stock return volatility (%)	11.00	11.00	11.00
Market price of risk	0.275	0.275	0.275
Equity premium (%)	3.03	3.03	3.03
Expected output growth (%)	1.43	0.05	2.35
Interest rate (%)	0.79	-1.66	3.09
Expected stock return (%)	3.81	1.37	6.12
Investment-capital ratio (%)	4.02	5.36	3.05
Tobin's q	2.52	5.09	1.85
Price-dividend ratio	42.04	109.53	26.57

Table 2: **Equilibrium outcomes under agreement:** The table reports the equlibrium outcomes in homeogeneous beliefs economies in which all investors have either the objective beliefs, the pessimistic beliefs, or the optimistic beliefs. As explained in the text, all equilibrium outcomes reported in the Table are constant in the homogeneous beliefs economies.

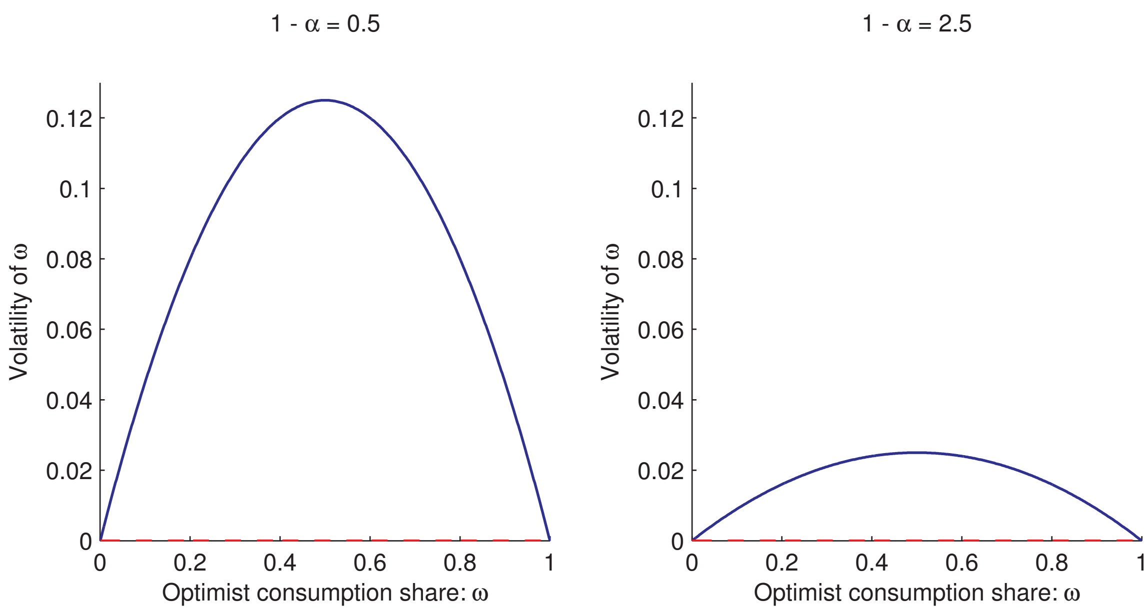

Figure 1: **Dynamics of optimist's consumption share, ω:** The Figure plots the volatility of the optimist's share of aggregate consumption ω_t against the current value of ω_t. The solid line is for the heterogenous beliefs economy and the dashed line is for the homogeneous beliefs economy where all investors have objective beliefs. The left plot has relative risk aversion of 0.5 and the right plot has relative risk aversion of 2.5 with the remainder of parameters used are reported in Table 1.

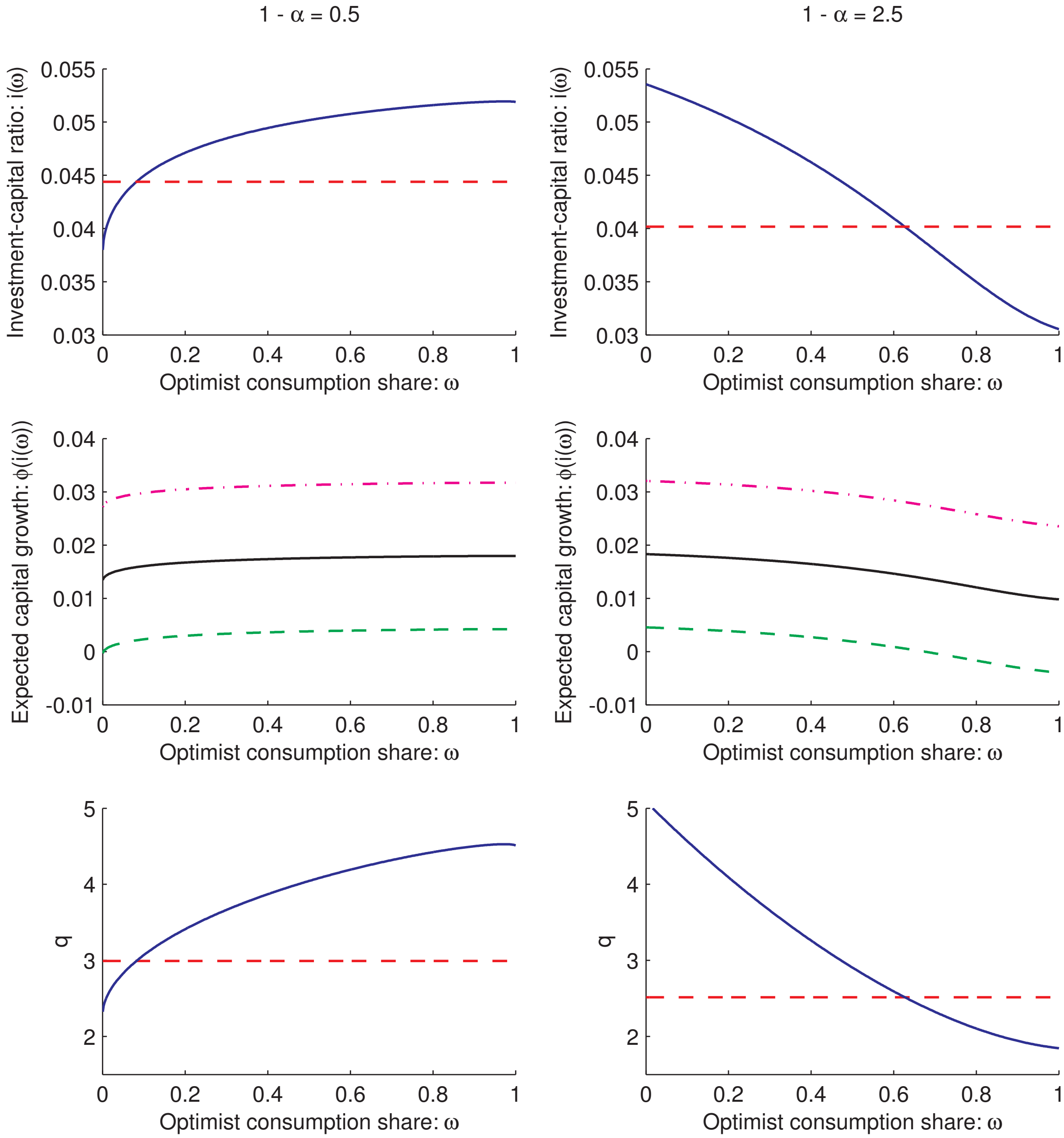

Figure 2: **Investment, expected capital growth and Tobin's q:** The top row of plots is the investment-capital ratio plotted against the optimist's consumption share ω, the middle row of plots is Tobin's q against ω and the bottom row of plots is the consumption-capital ratio against ω. In the top and bottom plot, the solid line is for the heterogeneous beliefs economy and the dashed line is for the homogeneous beliefs economy where all investors have objective beliefs. In the center row of plots the solid line corresponds to the objective (z) measure, the dashed-dotted line corresponds to the optimist's (a) measure and the dashed line corresponds to the pessimist's (b) measure. The left column of plots has relative risk aversion of 0.5 and the right plot has relative risk aversion of 2.5 with the remainder of parameters reported in Table 1.

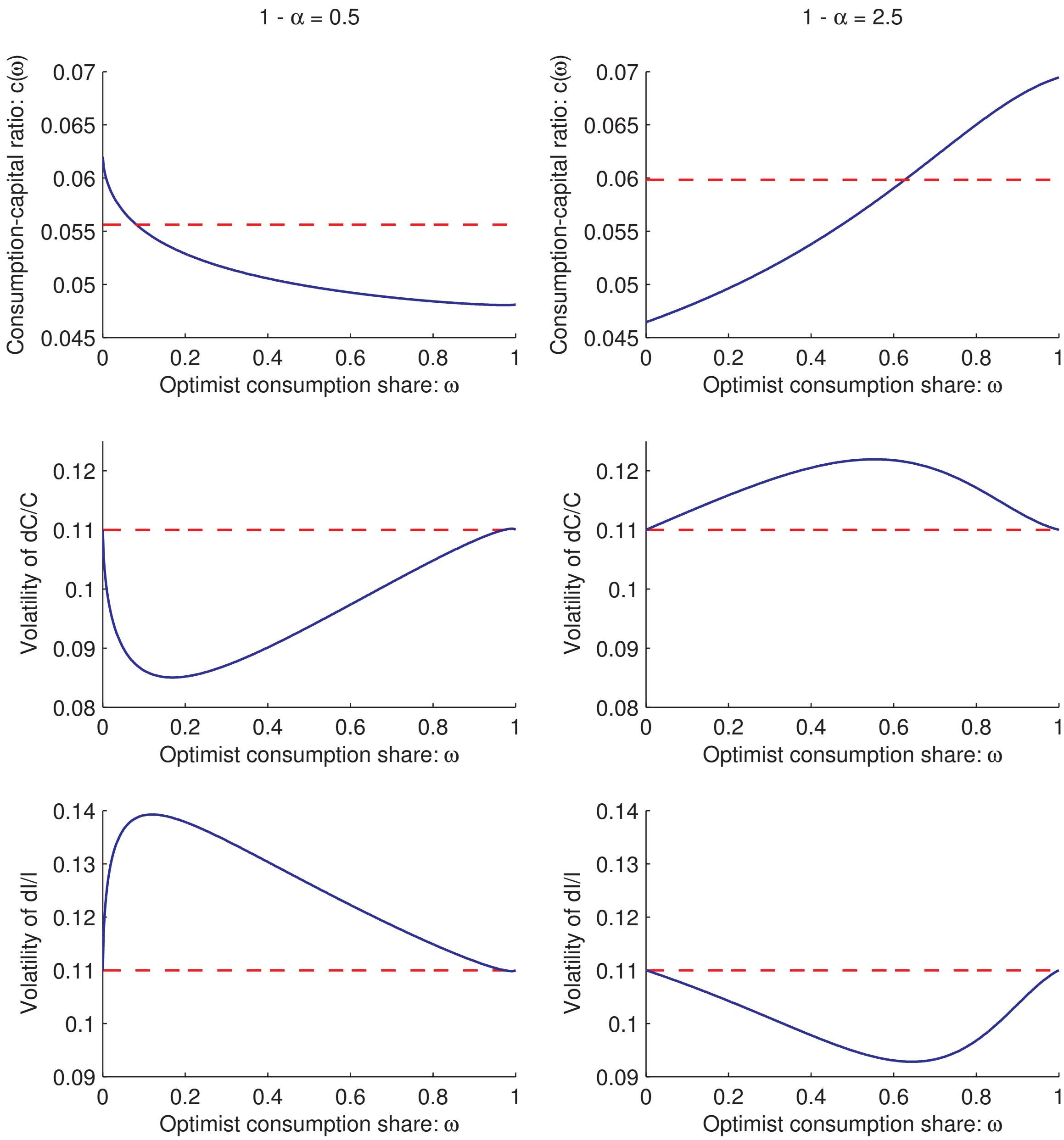

Figure 3: **Consumption growth and investment growth volatility:** The top row of plots is the consumption-capital ratio against the optimist's consumption share ω. The middle row of plots is the volatility of consumption growth plotted against ω, and the final row plots is the volatility of investment growth plotted against ω. In all the plots, the solid line is for the heterogeneous beliefs economy and the solid line is for the homogeneous beliefs economy with the objective beliefs. The left column of plots has relative risk aversion of 0.5 and the right plot has relative risk aversion of 2.5 with the remainder of parameters reported in Table 1.

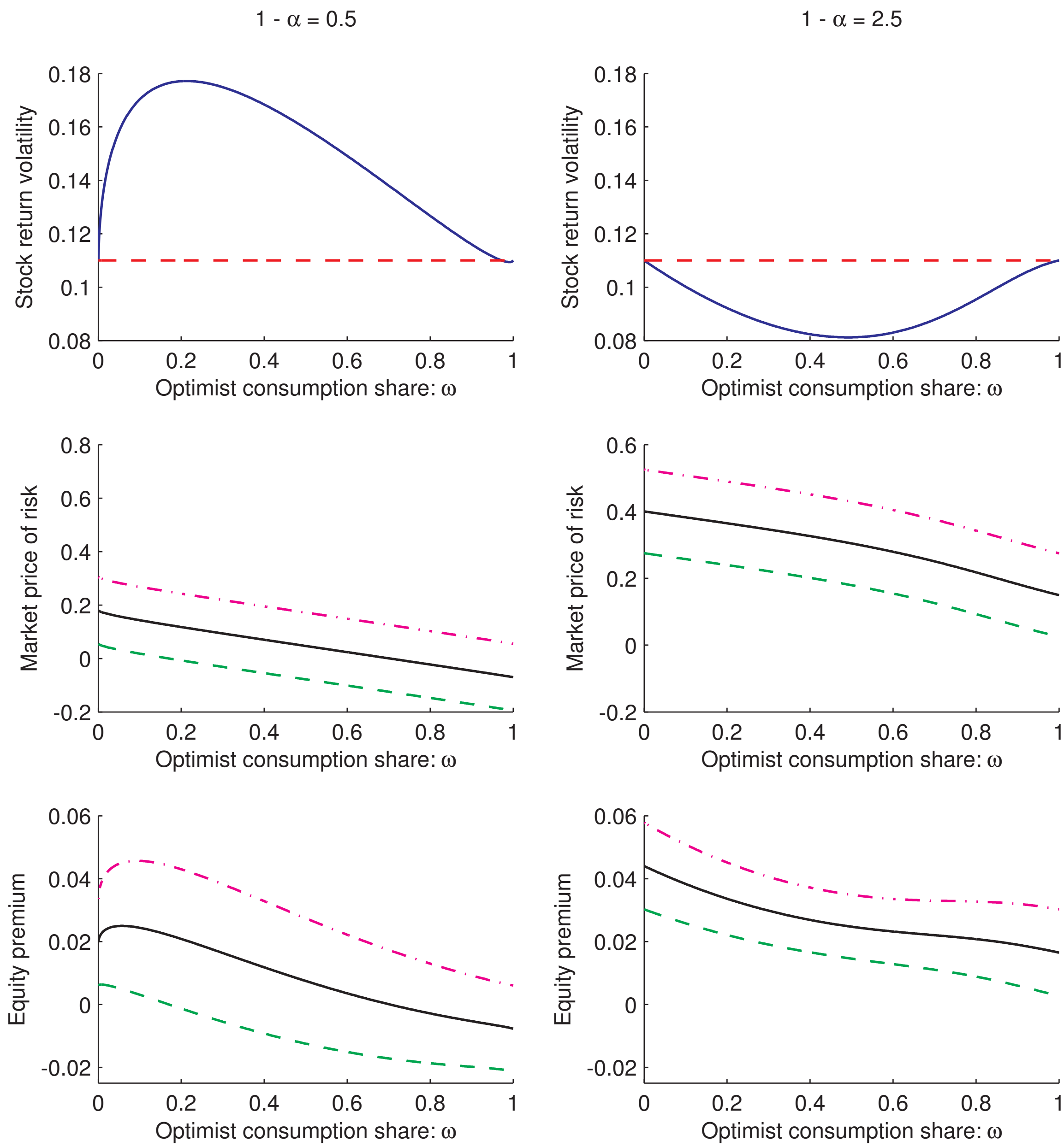

Figure 4: **Stock return volatility and the pricing of risk:** The top row of plots is the stock return volatility plotted against ω. The solid line is for the heterogeneous beliefs economy and the dashed line is for the homogeneous beliefs economy where all investors have the objective beliefs. The middle row of plots is the market price of risk plotted against ω. The bottom row of plots is the equity premium plotted against ω. In the middle and bottom rows of plots, the solid line is under the objective beliefs, the dashed line is under the pessimist's beliefs, and the dot-dashed line is under the optimist's beliefs. The left column of plots has relative risk aversion of 0.5 and the right plot has relative risk aversion of 2.5 with the remainder of parameters reported in Table 1.

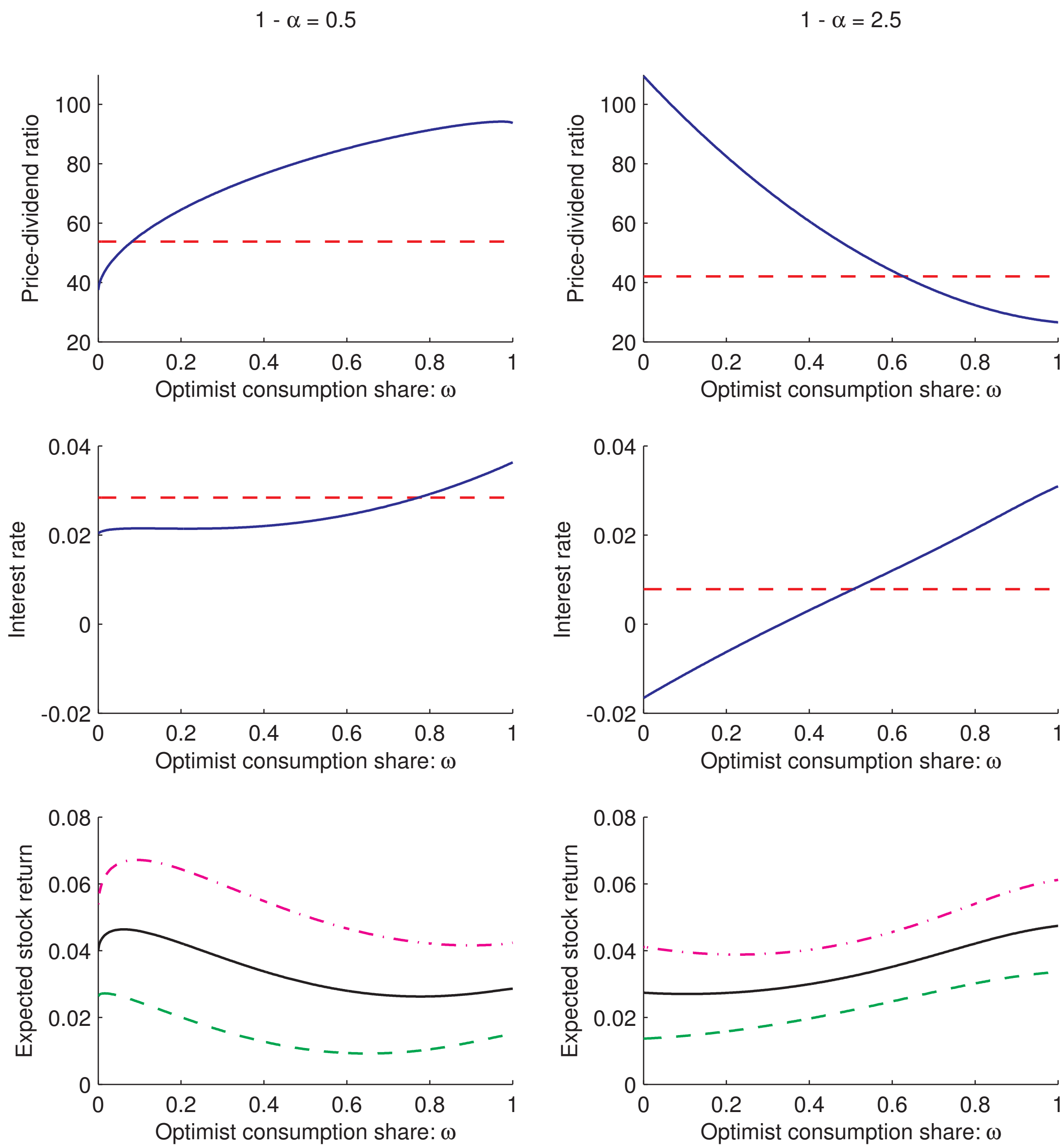

Figure 5: **Asset Prices:** The top row of plots is the Price-dividend ratio plotted against the optimist's consumption share ω. The solid line is for the heterogeneous beliefs economy and the dash line is for the homogeneous beliefs economy where all investors have the objective beliefs. The top row of plots is the locally riskless interest rate plotted against the optimist's consumption share ω. Here, the solid line is for the heterogeneous beliefs economy and the dashed line is for the homogeneous beliefs economy where all investors have the objective beliefs. The bottom row of plots is the expected stock return plotted against the optimist's consumption share ω. Here, the solid line is under the objective beliefs, the dashed line is under the pessimist's beliefs, and the dot-dashed line is under the optimist's beliefs. The left column of plots has relative risk aversion of 0.5 and the right plot has relative risk aversion of 2.5 with the remainder of parameters reported in Table 1.

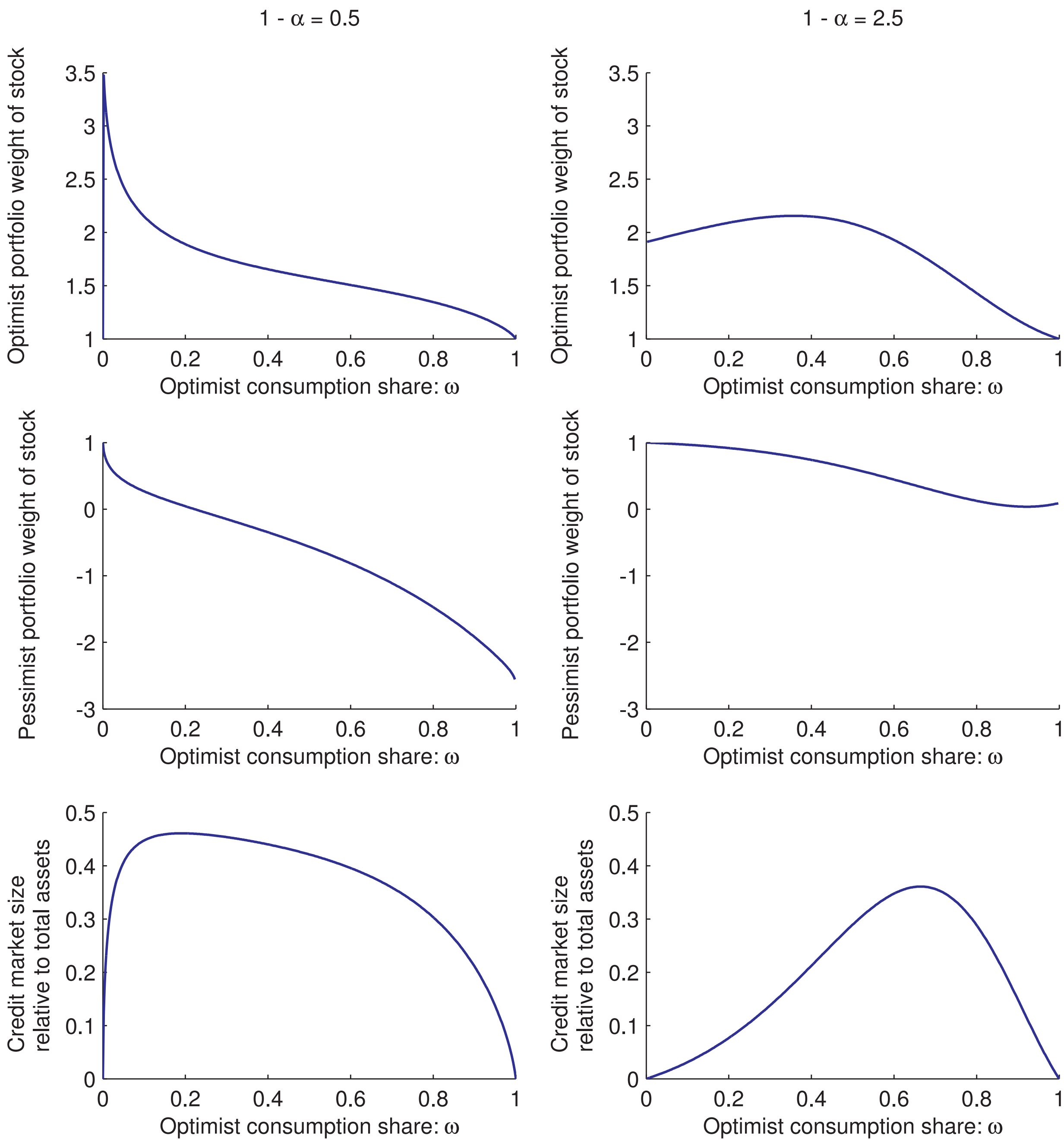

Figure 6: **Portfolios:** The top row of plots is the optimist's portfolio weight in the stock plotted against the optimist's consumption share ω. l The middle row in the plot is the pessimist's portfolio weight in the stock plotted against ω. The bottom row in the plot is the size of the credit market relative to the value of the stock market plotted against ω. The left column of plots has relative risk aversion of 0.5 and the right plot has relative risk aversion of 2.5 with the remainder of parameters reported in Table 1.

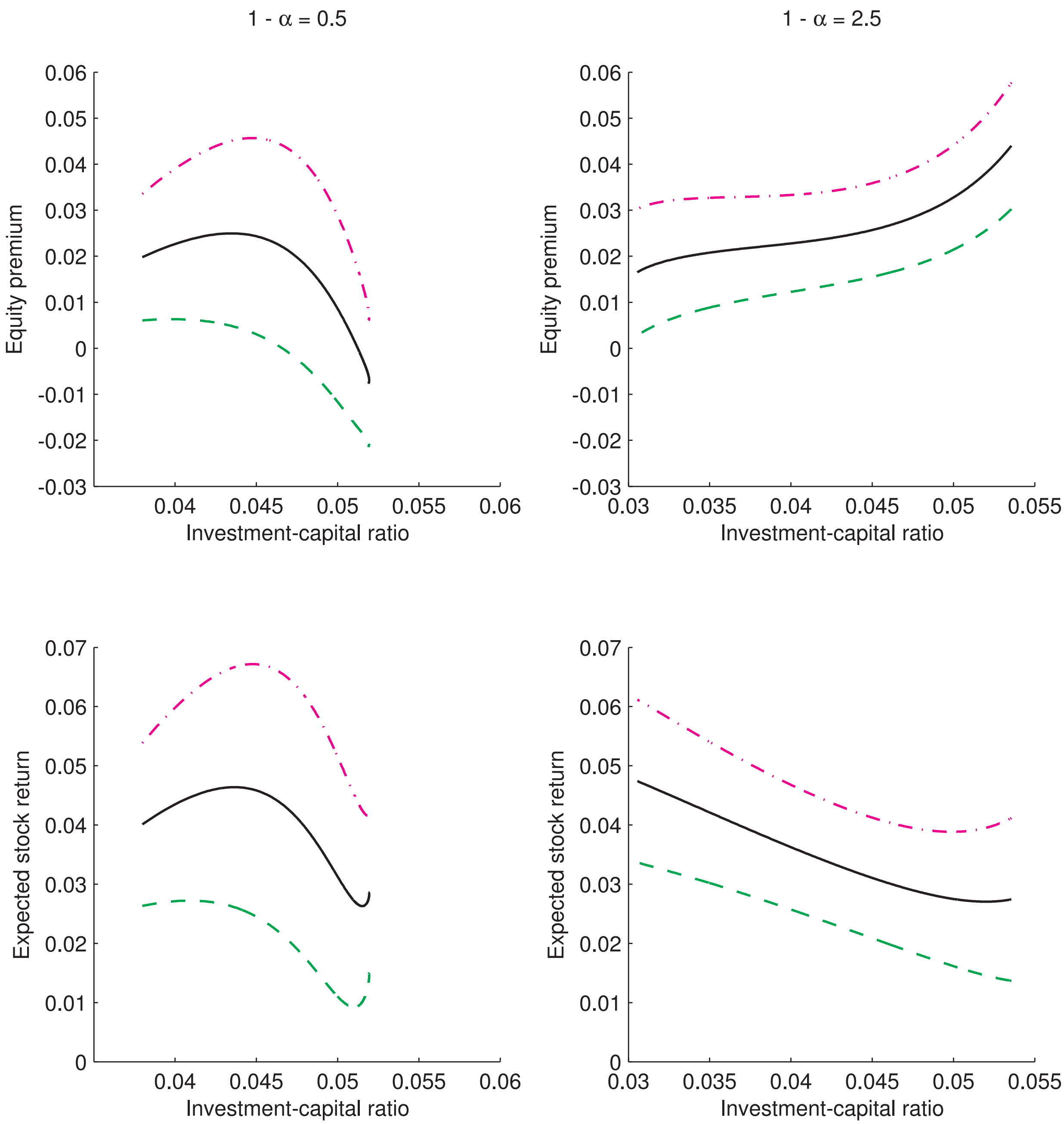

Figure 7: **Equity Returns and the investment-capital ratio:** The top row of plots is the equity premium under the objective, pessimistic and optimistic beliefs plotted against the investment-capital ratio and the bottom row of plots is the expected stock return under the objective, pessimistic and optimistic beliefs plotted against the investment-capital ratio. In all plots, the solid line is under the objective beliefs, the dashed line is under the pessimist's beliefs, and the dot-dashed line is under the optimist's beliefs. The left column of plots has relative risk aversion of 0.5 and the right plot has relative risk aversion of 2.5 with the remainder of parameters reported in Table 1.